Making Calculus Accessible

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Contents

Chapter 1

Preliminaries

1.1 Algebra and Trigonometry Review

You won't need everything you learned in Precalculus, but there are several topics which will be important. Review by working through the following problems.

Review Problems

1. Find an equation of a line in the form $y = mx + b$ for a line with slope -2 which passes through the point $(1, -6)$.
2. Find an equation of a line in the form $y = mx + b$ for a line passing through points $(3, -5)$ and $(-2, 1)$.
3. Simplify by factoring the numerator: $\dfrac{x^2 - x - 6}{x - 3}$.
4. Simplify by factoring the numerator: $\dfrac{2x^2 + 3x - 5}{x - 1}$.
5. Rationalize the denominator: $\dfrac{3}{\sqrt{7} - 2}$.
6. Rationalize the numerator: $\dfrac{\sqrt{x} - 1}{x - 1}$.
7. Rationalize the numerator: $\dfrac{\sqrt{x} - \sqrt{a}}{x - a}$.
8. Simplify: $(\sqrt{x})^4$.
9. Expand, combining exponents: $x^2(x + \sqrt{x})$.
10. Write using negative exponents: $\dfrac{2}{x^4} - \dfrac{3}{\sqrt{x}}$.
11. Write using positive exponents in the denominator: $x^{-1/3} + x^{-5}$.
12. Expand, combining exponents: $x^3 \left(\sqrt{x} + \dfrac{1}{\sqrt{x}} \right)$.
13. Factor out x^2 from the following expression: $3x^7 + x^{7/2}$.
14. Factor out $x^{3/2}$ from the following expression: $2x^4 - x^{3/2}$.
15. Add by finding a common denominator: $\dfrac{2}{5} + \dfrac{7}{6}$
16. Simplify using a common denominator: $\dfrac{1}{2x} - \dfrac{2}{y}$
17. Simplify: $\dfrac{\frac{1}{2} + \frac{1}{x}}{x}$

18. Simplify: $\dfrac{\frac{1}{3} - \frac{1}{y}}{3 - y}$

19. Convert from degrees to radians:

    (a) $60°$

    (b) $180°$

    (c) $-210°$

20. Convert from radians to degrees:

    (a) $\dfrac{3\pi}{4}$

    (b) $-\dfrac{\pi}{2}$

    (c) $\dfrac{5\pi}{3}$

21. Evaluate the following, giving exact answers.

    (a) $\cos 0$

    (b) $\sin 90°$

    (c) $\tan \dfrac{\pi}{3}$

    (d) $\sin \dfrac{5\pi}{4}$

    (e) $\cos 300°$

    (f) $\tan \dfrac{3\pi}{2}$

Assessment Expectations: For algebra problems, any like those above. For the Unit Circle, you will be given a blank Unit Circle, with questions like:

1. What angle in degrees corresponds to Point A?
2. What angle in radians corresponds to Point B?
3. What are the coordinates of Point C?
4. What is $\sin(60°)$? Can be cos or tan, and the angle may be in radians.

Solutions

1.

$$\begin{aligned} y &= mx + b \\ y &= -2x + b \\ -6 &= -2(1) + b \\ -4 &= b \\ y &= -2x - 4 \end{aligned}$$

2. To find the slope, use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, with $(x_1, y_1) = (3, -5)$ and $(x_2, y_2) = (-2, 1)$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - (-5)}{-2 - 3} \\ &= -\frac{6}{5} \end{aligned}$$

Now use $y = mx + b$ with the point $(3, -5)$.

$$\begin{aligned} y &= -\frac{6}{5}x + b \\ -5 &= -\frac{6}{5}(3) + b \\ -\frac{25}{5} + \frac{18}{5} &= b \\ -\frac{7}{5} &= b \\ y &= -\frac{6}{5}x - \frac{7}{5} \end{aligned}$$

3.

$$\begin{aligned} \frac{x^2 - x - 6}{x - 3} &= \frac{(x-3)(x+2)}{x-3} \\ &= x + 2 \end{aligned}$$

4.

$$\begin{aligned}\frac{2x^2+3x-5}{x-1} &= \frac{(x-1)(2x+5)}{x-1}\\ &= 2x+5\end{aligned}$$

5.

$$\begin{aligned}\frac{3}{\sqrt{7}-2} &= \frac{3}{\sqrt{7}-2}\cdot\frac{\sqrt{7}+2}{\sqrt{7}+2}\\ &= \frac{3(\sqrt{7}+2)}{7+2\sqrt{7}-2\sqrt{7}-4}\\ &= \frac{3(\sqrt{7}+2)}{3}\\ &= \sqrt{7}+2\end{aligned}$$

6.

$$\begin{aligned}\frac{\sqrt{x}-1}{x-1} &= \frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}\\ &= \frac{x+\sqrt{x}-\sqrt{x}-1}{(x-1)(\sqrt{x}+1)}\\ &= \frac{x-1}{(x-1)(\sqrt{x}+1)}\\ &= \frac{1}{\sqrt{x}+1}\end{aligned}$$

7.

$$\begin{aligned}\frac{\sqrt{x}-\sqrt{a}}{x-a} &= \frac{\sqrt{x}-\sqrt{a}}{x-a}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\\ &= \frac{x+\sqrt{x}\sqrt{a}-\sqrt{x}\sqrt{a}-a}{(x-a)(\sqrt{x}+\sqrt{a})}\\ &= \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}\\ &= \frac{1}{\sqrt{x}+\sqrt{a}}\end{aligned}$$

8.

$$\begin{aligned}(\sqrt{x})^4 &= \left(x^{1/2}\right)^4\\ &= x^{(1/2)4}\\ &= x^2\end{aligned}$$

9.

$$\begin{aligned}x^2(x+\sqrt{x}) &= x^2(x+x^{1/2})\\ &= x^2\cdot x + x^2\cdot x^{1/2}\\ &= x^3+x^{5/2}\end{aligned}$$

10.

$$\frac{2}{x^4} - \frac{3}{\sqrt{x}} = 2x^{-4} - 3x^{-1/2}$$

11.

$$x^{-1/3} + x^{-5} = \frac{1}{\sqrt[3]{x}} + \frac{1}{x^5}$$

12.

$$\begin{aligned} x^3\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) &= x^3(x^{1/2} + x^{-1/2}) \\ &= x^3 \cdot x^{1/2} + x^3 \cdot x^{-1/2} \\ &= x^{7/2} + x^{5/2} \end{aligned}$$

13.

$$\begin{aligned} 3x^7 + x^{7/2} &= x^2(3x^{7-2} + x^{7/2-2}) \\ &= x^2(3x^5 + x^{3/2}) \end{aligned}$$

14.

$$\begin{aligned} 2x^4 - x^{3/2} &= x^{3/2}(2x^{4-3/2} - x^{3/2-3/2}) \\ &= x^{3/2}(2x^{5/2} - 1) \end{aligned}$$

15.

$$\begin{aligned} \frac{2}{5} + \frac{7}{6} &= \frac{2}{5} \cdot \frac{6}{6} + \frac{7}{6} \cdot \frac{5}{5} \\ &= \frac{12}{30} + \frac{35}{30} \\ &= \frac{47}{30} \end{aligned}$$

16.

$$\begin{aligned} \frac{1}{2x} - \frac{2}{y} &= \frac{1}{2x} \cdot \frac{y}{y} - \frac{2}{y} \cdot \frac{2x}{2x} \\ &= \frac{y}{2xy} - \frac{4x}{2xy} \\ &= \frac{y - 4x}{2xy} \end{aligned}$$

17. Method 1: First, combine the numerator.

$$\begin{aligned} \frac{1}{2} + \frac{1}{x} &= \frac{1}{2} \cdot \frac{x}{x} + \frac{1}{x} \cdot \frac{2}{2} \\ &= \frac{x + 2}{2x} \end{aligned}$$

Then multipliy by the reciprocal of the denominator.

$$\frac{\frac{x+2}{2x}}{x} = \frac{x+2}{2x} \cdot \frac{1}{x} = \frac{x+2}{2x^2}$$

Method 2: The least common denominator in the fractions is $2x$. Multiply top and bottom by $2x$.

$$\frac{\frac{1}{2}+\frac{1}{x}}{x} = \frac{\frac{1}{2}+\frac{1}{x}}{x} \cdot \frac{2x}{2x} = \frac{\frac{1}{2}\cdot 2x + \frac{1}{x}\cdot 2x}{x \cdot 2x} = \frac{x+2}{2x^2}$$

18. Method 1: First, combine the numerator.

$$\frac{1}{3} - \frac{1}{y} = \frac{1}{3}\cdot\frac{y}{y} - \frac{1}{y}\cdot\frac{3}{3} = \frac{y-3}{3y}$$

Then multipliy by the reciprocal of the denominator.

$$\frac{\frac{y-3}{3y}}{y} = \frac{y-3}{3y} \cdot \frac{1}{3-y} = \frac{y-3}{3y(3-y)} = \frac{-(3-y)}{3y(3-y)} = -\frac{1}{3y}$$

Method 2: The least common denominator in the fractions is $3y$. Multiply top and bottom by $3y$.

$$\frac{\frac{1}{3}-\frac{1}{y}}{y} = \frac{\frac{1}{3}-\frac{1}{y}}{3-y} \cdot \frac{3y}{3y} = \frac{\frac{1}{3}\cdot 3y - \frac{1}{y}\cdot 3y}{(3-y)\cdot 3y} = \frac{y-3}{(3-y)\cdot 3y} = \frac{-(3-y)}{(3-y)\cdot 3y} = -\frac{1}{3y}$$

19. Use the conversion factor 1 degree $= \dfrac{\pi}{180}$ radians.

   (a) $60 \cdot \dfrac{\pi}{180} = \dfrac{\pi}{3}$
   (b) $180 \cdot \dfrac{\pi}{180} = \pi$
   (c) $-210 \cdot \dfrac{\pi}{180} = -\dfrac{7\pi}{6}$

20. Use the converstion factor 1 radian $= \dfrac{180}{\pi}$ degrees.

   (a) $\dfrac{3\pi}{4} \cdot \dfrac{180}{\pi} = 135°$
   (b) $-\dfrac{\pi}{2} \cdot \dfrac{180}{\pi} = -90°$
   (c) $\dfrac{5\pi}{3} \cdot \dfrac{180}{\pi} = 300°$

21. Use the unit circle. If $\cos\theta = 0$, then $\tan\theta$ is undefined. Otherwise, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

   (a) 1
   (b) 1
   (c) $\sqrt{3}$
   (d) $-\dfrac{1}{\sqrt{2}}$
   (e) $\dfrac{1}{2}$
   (f) Undefined

1.2 Introduction to Physics

Much of calculus was developed to study physics. While a course in physics is not required for calculus, there are a few fundamental concepts from physics that we will use over and over again. This is a summary of those concepts.

Most of us are familiar with driving a car. The speedometer measures the **speed** at which you're traveling, while the odometer measures the **distance** you've traveled.

Let's suppose you take a two-hour drive, and you drive at a constant rate of 30 km/hr. Below is a graph of your starting position, at $x = 0$. What is your ending position?

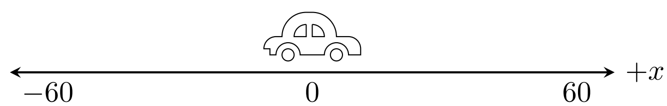


Figure 1.1: Starting position.

If you drove east for two hours (we'll describe going in the positive direction as going east, and going in the negative direction as going west), you'd be 60 km east of where you began.

Figure 1.2: Ending position A.

But if you drove west for two hours, you'd be 60 km west of where you began.

Figure 1.3: Ending position B.

But maybe you drove east for one hour, and then turned around and drove west for an hour. Then you'd be right back where you started.

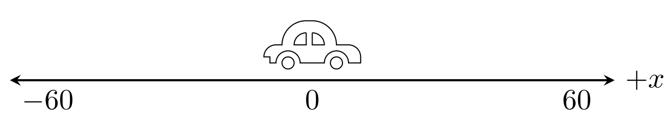


Figure 1.4: Ending position C.

Since you're driving at a constant speed, your speed graph would look like this.

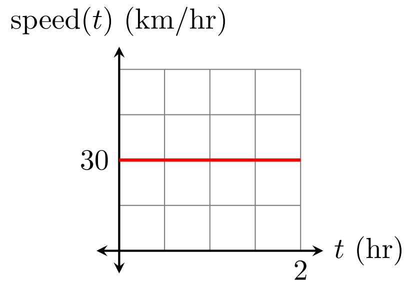


Figure 1.5: Speed graph.

The main issue is this: you can't know your ending position by looking at the speed graph. There is not enough information. That is why the concept of **velocity** is so important in science. Essentially, velocity is speed *and* direction. When you're driving east, your speed and velocity are both 30 km/hr. But when you're driving west, your speed is 30 km/hr, but your velocity is -30 km/hr.

What about a graph of the distance you traveled? Since distance equals rate times time and you're driving at a constant speed, you've traveled $30 \times t$ km in t hours. So the distance traveled up to time t is represented by the graph below.

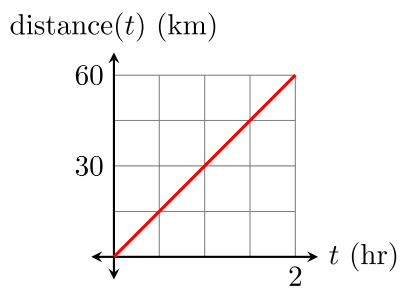


Figure 1.6: distance(t) = $30t$

This graph tells you that you've driven a total of 60 km, but there is no way to know *where* you ended up. Where you end up relative to where you began is called **displacement** in physics. Similar to how velocity is speed with a direction, displacement is *distance* with a direction. So your displacement in Ending position A is 60 km, but your displacement in Ending position B is -60 km, since you ended up 60 km west of where you started. Your displacement in Ending position C is 0 km, since you're back where you started.

Ending position A

Let's see how we apply the concepts of velocity and displacement to each of the three scenarios described above. We'll redraw Figure 1.2 with an arrow representing the path taken.

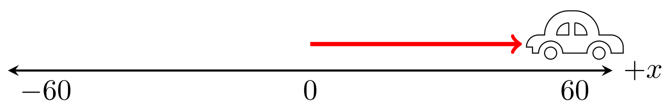


Figure 1.7: Ending position A.

In this scenario, speed and velocity are the same: 30 km/hr for two hours. The letter "v" is used in physics to represent velocity. This is shown on the left of Figure 1.8.

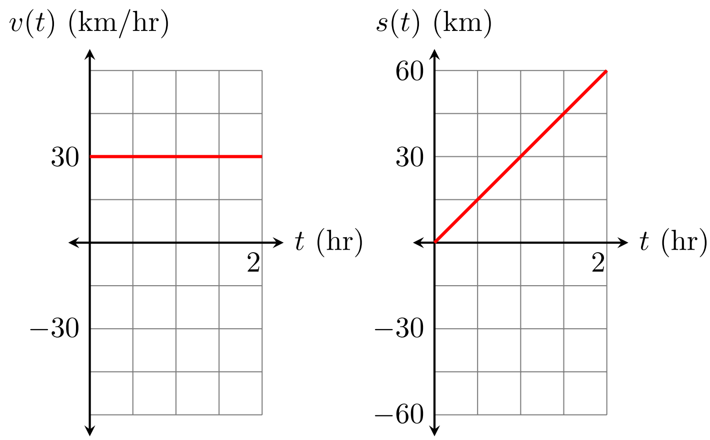


Figure 1.8: Velocity and displacement graphs for ending position A.

In this case, the distance is the same as the displacement, so the displacement graph is identical to Figure 1.6. We use the letter "s" for displacment, since the letter "d" is used for describing derivatives in calculus. Notice that the velocity is positive here, and the slope of the displacement is positive. This is not a coincidence, and is another concept we'll be exploring in calculus.

Ending position B

Let's redraw Figure 1.3 to reflect the path taken.

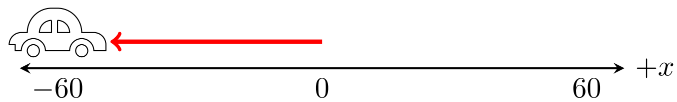


Figure 1.9: Ending position B.

In this case, you drove west for two hours, and so your velocity is −30 km/hr for two hours. This is shown on the left of Figure 1.10. Because your velocity is negative, you end up 60 km west from where you started: a total displacement of −60 km. Notice that the displacement graph has a negative slope because the velocity is negative. This is shown on the right of Figure 1.10.

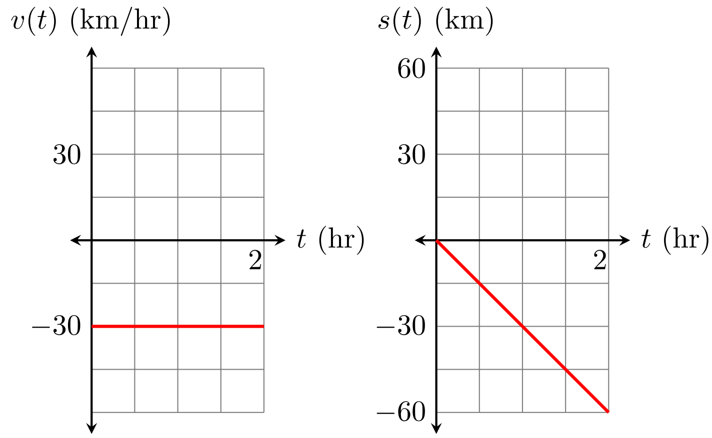


Figure 1.10: Velocity and displacement graphs for ending position B.

Ending position C

Here, Figure 1.4 is redrawn to include the path taken.

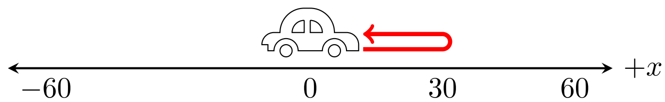


Figure 1.11: Ending position C.

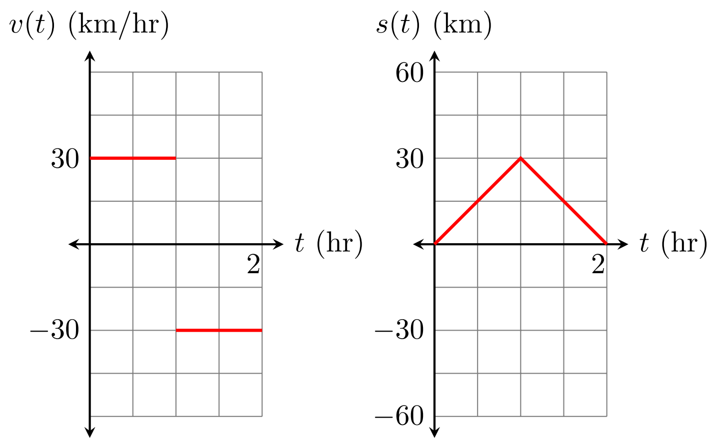


Figure 1.12: Velocity and displacement graphs for ending position C.

Here, you drove east for one hour (30 km/hr) and west for the next hour (−30 km/hr). So the velocity curve jumps down to −30 after one hour.

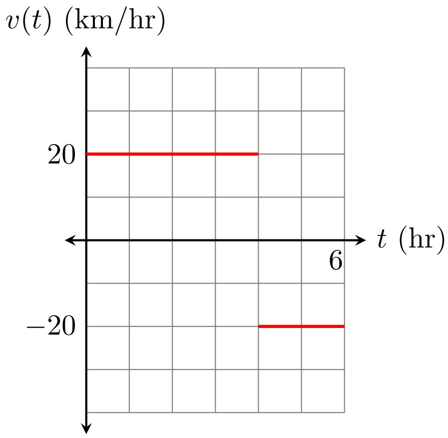
But for ending position C, you start driving east, so the displacement graph is sloping upward. But after an hour – since you turned around – the displacement graph begins sloping downward, so by the time two hours have gone by, your displacement is 0 km, since you ended where you started.

The important point is this: if I gave you one of the *velocity* graphs for any of the three scenarios, you could tell me *exactly* where I ended up. But all three ending positions have the *same* speed graph (shown in Figure 1.5). So in physics and science, "velocity" is a much more useful concept than "speed."

Also, if I gave you one of the displacement graphs for any of these scenarios, you could tell me *exactly* what my trip looked like and where I ended up. But if I just gave you the distance graph (as shown in Figure 1.6), the *only* thing you could tell me was that I drove 60 km. There would not be enough information to conclude any more about the nature of my trip.

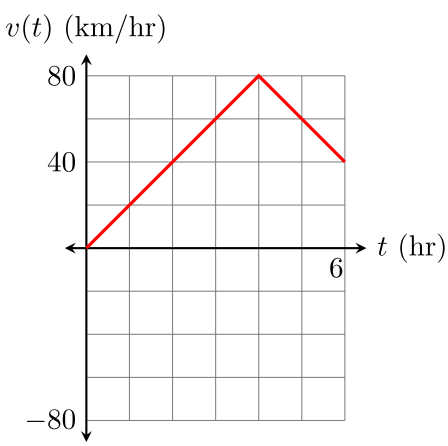
Example 1

Below is the velocity graph for a trip. Describe the journey and draw the corresponding displacement graph.



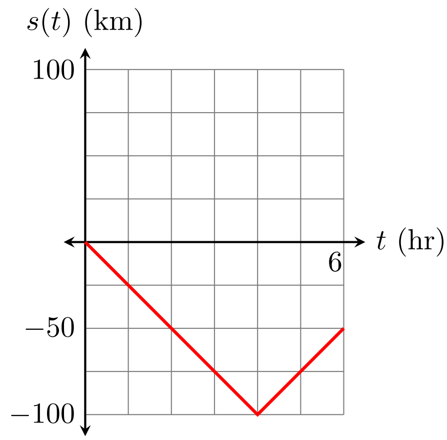
For this journey, we are driving east at 20 km/hr for four hours, then turning around and driving west at 20 km/hr for two more hours. What does the displacement graph look like?

After 4 hours at 20 km/hr, you would have gone 80 km. So this tells you how to draw the first half of the displacement graph. Once you've gone 80 km, you turn around and drive at -20 km/hr for two hours, so you go west 40 km, bringing your net displacement to 40 km. This tells you how to draw the second part of the graph.



Example 2

Below is the displacement graph for a trip. Describe the journey and draw the corresponding velocity graph.



For this journey, we're driving 100 km west, and then 50 km east, so that we end up 50 km west of where we started. What does the velocity graph look like?

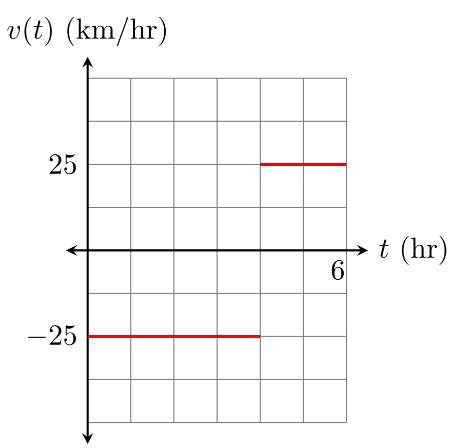
Recall that the velocity is the slope of the displacement graph, so for the first four hours, we get

$$v(t) = \frac{\text{rise}}{\text{run}} = \frac{-100}{4} = -25 \text{ km/hr}.$$

For the last two hours, we get

$$v(t) = \frac{\text{rise}}{\text{run}} = \frac{50}{2} = 25 \text{ km/hr}.$$

Thus, the velocity graph looks like the graph below.

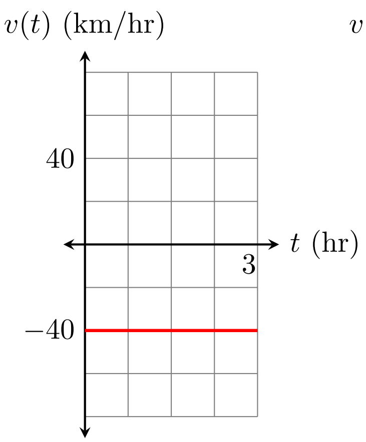


Summary

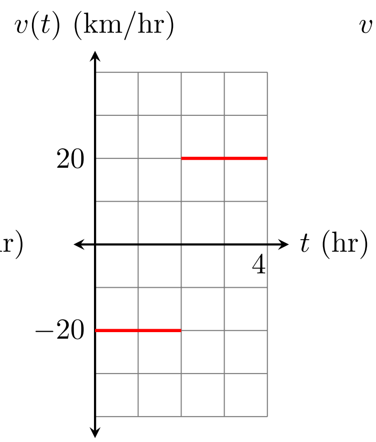
While speed and distance are concepts well-suited to day-to-day life, in the world of physics and science, they are not very precise. Graphs of speed and distance provide very little information about the nature of a journey. However, by introducing the concepts of velocity and displacement, we get an extremely accurate representation of what is actually going on. Essentially, calculus is an in-depth study of the relationship between velocity and displacement graphs.

Homework

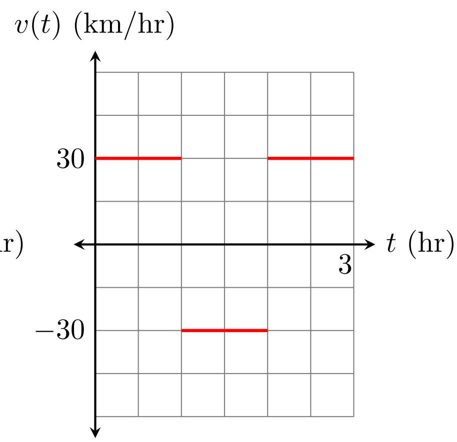
1. Each of the following is the velocity graph of a car trip. For each graph, (1) write a sentence explaining the trip in words, and (2) draw the corresponding displacement graph. Label your graphs carefully!



(a)

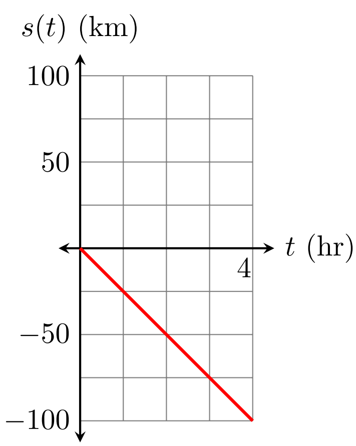


(b)

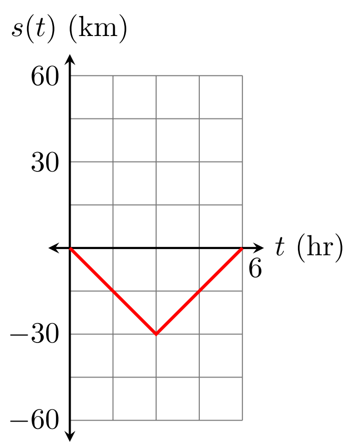


(c)

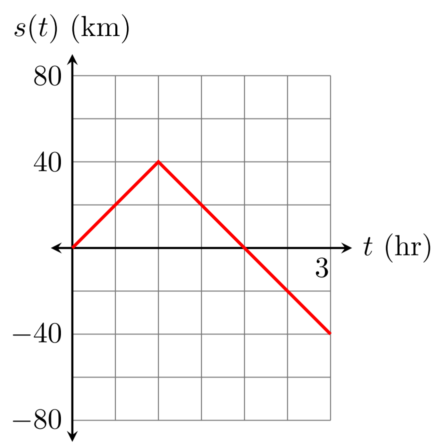
2. Each of the following is the displacement graph of a car trip. For each graph, (1) write a sentence explaining the trip in words, and (2) draw the corresponding velocity graph. Label your graphs carefully!



(a)



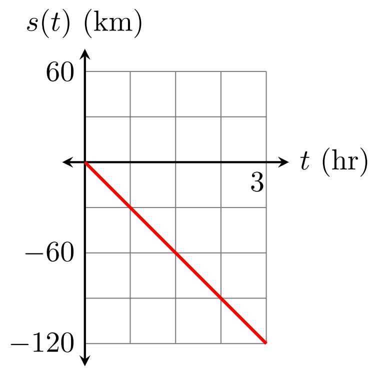
(b)



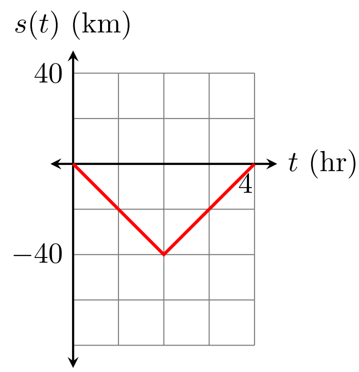
(c)

Solutions

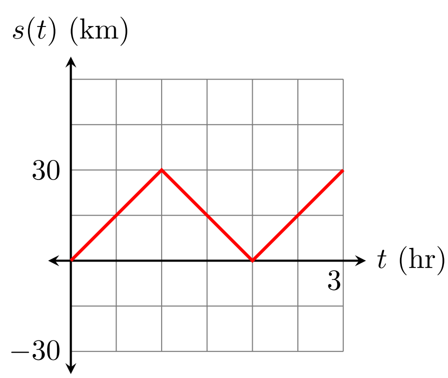
1. (a) You drove 40 km/hr west for three hours.



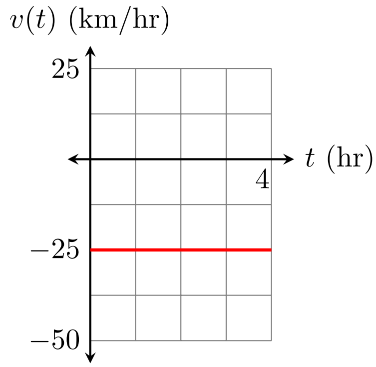
(b) You drove west at 20 km/hr for two hours, and then turned around and drove east at 20 km/hr for two hours.



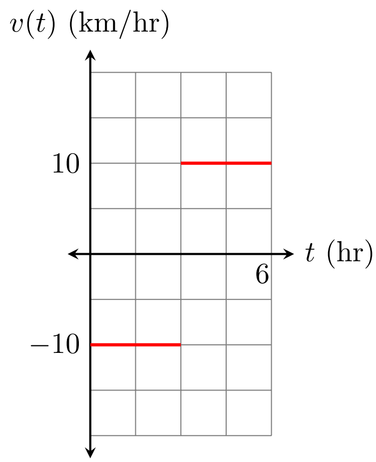
(c) You drove east at 30 km/hr for one hour, turned around and drove west at 30 km/hr for another hour, and then turned around and drove east for one hour at 30 km/hr.



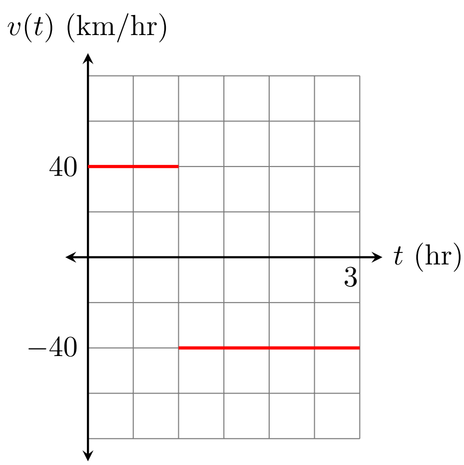
2. (a) You drove 100 km west in four hours, and therefore drove at a velocity of −25 km/hr.



(b) You drove 30 km west in three hours – at a velocity of −10 km/hr, and then drove east 30 km in three hours – at a velocity of 10 km/hr.



(c) You drove 40 km east in one hour (40 km/hr), and then turned around and drove 80 km west in two hours (a velocity of −40 km/hr).



1.3 What Calclulus is All About

We will look at the basic concepts behind calculus by studying three examples. Much of calculus comes from physics, so we'll focus our attention on velocity and displacement. In physics, velocity and displacement can be positive or negative, so we use these terms instead of speed and distance.

Example 1

The graphs below represent a road trip you might go on. You drive at a constant rate of 20 km/hr for 4 hours, and so $v(t) = 20$.

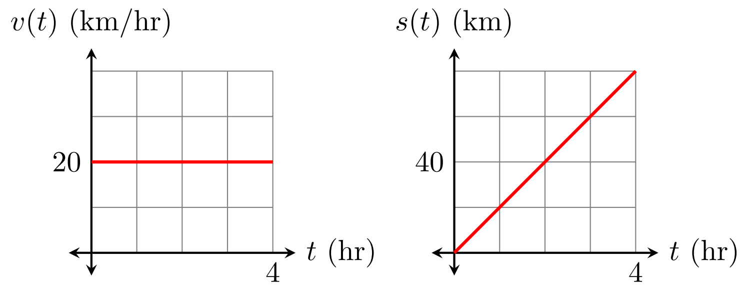


Figure 1.13: Velocity graph (left), and displacement graph (right).

Since displacement = rate × time, then $s(t) = 20t$. This is graphically represented below by the blue rectangle.

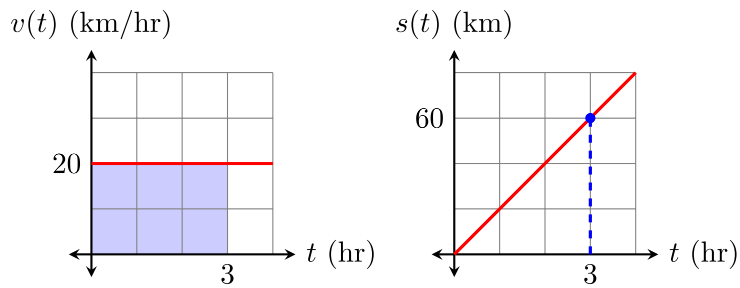


Figure 1.14: Velocity graph (left), and displacement graph (right).

Let's look at what happens after driving for three hours. You've traveled a total of $s(3) = 60$ km, which is represented by the blue rectangle; that is, the area under $v(t)$ up to three hours. Also, the slope of the line at $(3, 60)$ on the displacement curve is $v(3)$, which is 20 km/hr. It might look like the slope is 1, but remember that the units on the axes are different.

This example (and the ones that follow) illustrate this very important principle in physics:

> The area underneath the velocity curve up to time t corresponds to the displacement at time t, and the slope of the displacement curve at time t is the velocity at time t.
>
> Using function notation, we would say that the area up to t underneath $v(t)$ is $s(t)$, and the slope of the tangent line at $(t, s(t))$ is given by $v(t)$.

Example 2

Now we'll take a look at a road trip where your velocity is not constant. Since you are driving at 40 km/hr after 4 hours, your velocity is given by $v(t) = 10t$.

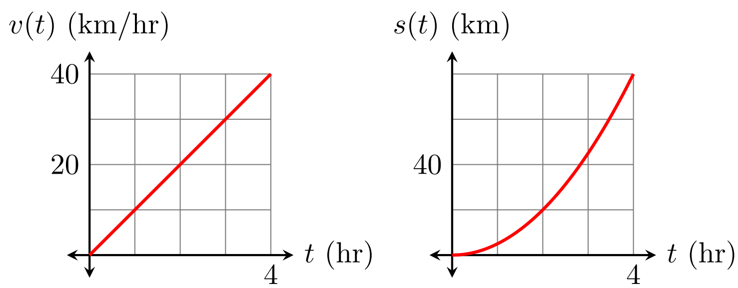


Figure 1.15: Velocity graph (left), and displacement graph (right).

Since your velocity is not constant, we can't use displacement = rate × time. But we can still use the fact that $s(t)$ is the area under $v(t)$ up to t hours, just as in Example 1.

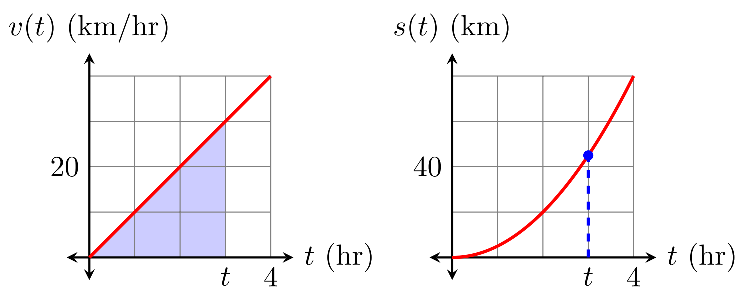


Figure 1.16: Velocity graph (left), and displacement graph (right).

Since the blue area is a triangle, we can use $A = \frac{1}{2}bh$, where $b = t$ is the base and $h = 10t$ is the height. So

$$\begin{aligned} s(t) &= \frac{1}{2} \cdot b \cdot h \\ &= \frac{1}{2} \cdot t \cdot 10t \\ &= 5t^2. \end{aligned}$$

This function is graphed on the right; it is a parabola.

In Example 1, we saw that the slope of the line at $(3, s(3))$ was 20 km/hr. In this example, $s(3) = 45$. But what is the slope of the parabola at $(3, 45)$? Now we've entered calculus territory. We're not only interested in the slope of a *line,* we're interested in the slope of a *curve.*

We can accomplish this by looking at the tangent line to a curve. Below (middle graph), you can see that the blue line intersects the parabola *only* at the point $(3, 45)$. In geometry, when a line intersects a curve at just one point, we call this a **tangent line.** When you zoom in on the blue box (right graph), you see that near the point $(3, 45)$, it's hard to tell the difference between the tangent line and the parabola. This is an important property in calculus.

What is the slope of this tangent line? Since $v(3) = 30$, this tangent line has a slope of 30 km/hr.

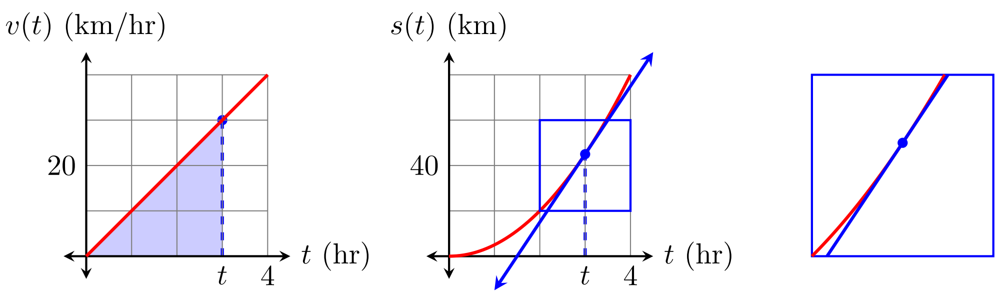


Figure 1.17: Velocity graph (left), tangent line (middle), zooming in (right).

Example 3

Let's look at another road trip. This time, you start out at 20 km/hr, but you slow down at a constant rate. After 2 hours, you turn around and start driving in the opposite direction. This is why we use *velocity* instead of *speed.* If you are driving east with a positive velocity, it means that if you turn around and drive west, your velocity is negative.

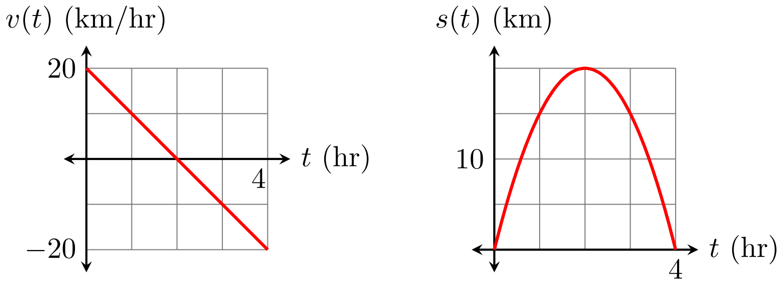


Figure 1.18: Velocity graph (left), displacement graph (right).

Since you start out at 20 km/hr and end up at -20 km/hr after 4 hours (meaning you end up driving in the opposite direction), then $v(t) = 20 - 10t$, which you get by finding the equation of the line between $(0, 20)$ and $(4, -20)$.

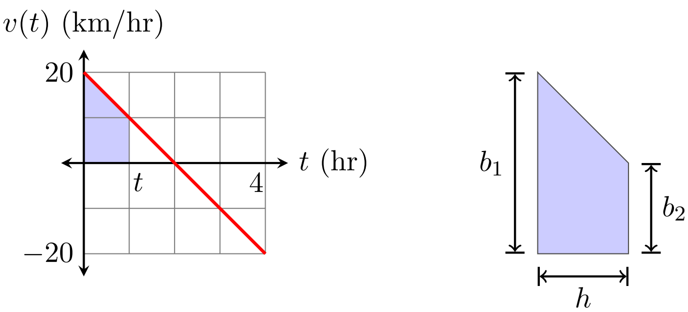


Figure 1.19: Area under the velocity graph (left), area of the trapezoid (enlarged, right).

As in Example 2, we can find the displacement curve by looking at an area. This time, the area is a trapezoid, so we need the formula $A = \dfrac{1}{2}(b_1 + b_2)h$ from geometry. In our example, b_1 is always 20, b_2 corresponds to $v(t)$, and h is just t. This means that

$$\begin{aligned} s(t) &= \frac{1}{2} \cdot (b_1 + b_2) \cdot h \\ &= \frac{1}{2} \cdot (20 + v(t)) \cdot t \\ &= \frac{1}{2}(20 + (20 - 10t)) \cdot t \\ &= 20t - 5t^2. \end{aligned}$$

Now look at the graph below. Since $v(t)$ is positive here, the slope of the tangent line to the curve $s(t)$ is positive.

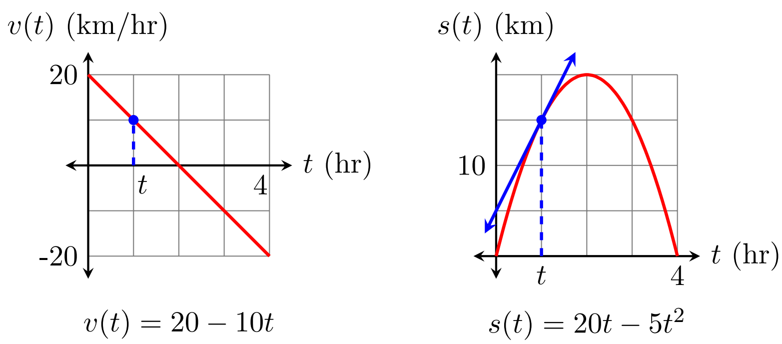


Figure 1.20: Velocity graph (left), displacement graph (right).

But in the next graph, the value of $v(t)$ is negative, and you can see that the slope of the tangent line to the curve is negative.

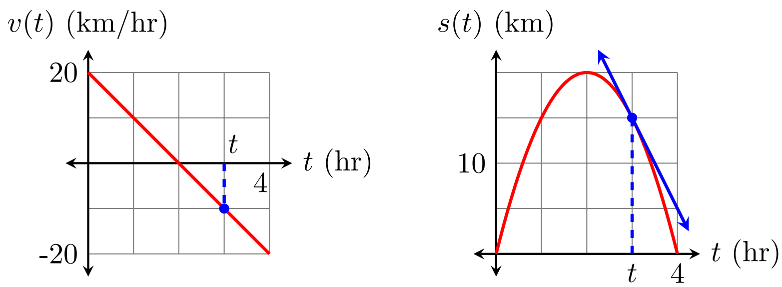


Figure 1.21: Velocity graph (left), displacement graph (right).

Now when $t = 4$, you can see that $s(4) = 0$. This represents the area under the velocity curve up to $t = 4$. How can this area be zero?

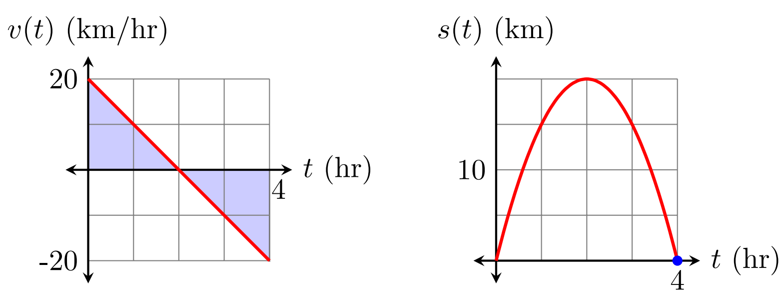


Figure 1.22: Velocity graph (left), displacement graph (right).

In calculus, areas can be *negative.* Let's say when your velocity is positive, you're driving east, so when your velocity is negative, you're driving west. Now the area *below* the velocity curve when the velocity is *positive* is how far you traveled east. But the area *above* the velocity curve when the velocity is *negative* is how far you traveled west – and this area is *negative.* Looking at the graph on the left, the triangles are congruent, but one has positive area and the other has negative area. Their areas cancel out, which is why your total displacement is 0 – you've traveled just as far while driving east as you did while driving west.

So considering velocity and displacement, instead of speed of distance, is very important in calculus.

When your velocity is positive, the corresponding slope of the tangent line on the displacement curve will be positive.

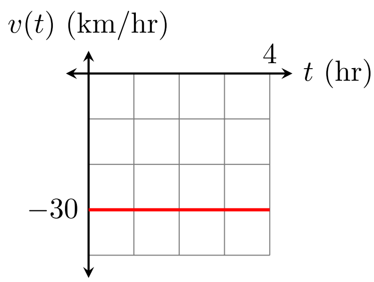
When your velocity is negative, you're traveling in the opposite direction, and the corresponding slope of the tangent line on the displacement curve will be negative.

The area corresponding to traveling with a positive velocity will always be positive, so your displacement will be positive.

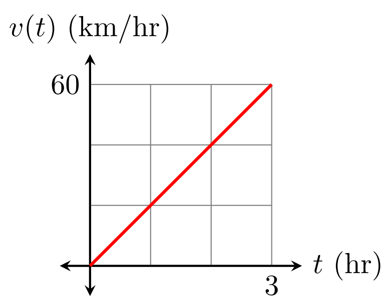
The area corresponding to traveling with a negative velocity in the opposite direction will always be negative, so your displacement will be negative.

Homework

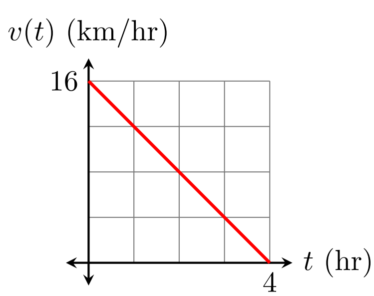
1. Below is a graph of a velocity curve. Find an equation for the displacement curve.



2. Below is a graph of a velocity curve. Find an equation for the displacement curve.



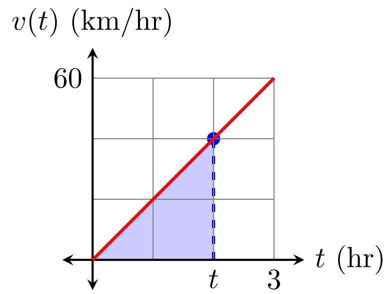
3. Below is a graph of a velocity curve. Find an equation for the displacement curve.



Solutions

1. Since the velocity is constant, we find displacement by multiplying the velocity by time, and so $s(t) = -30t$.

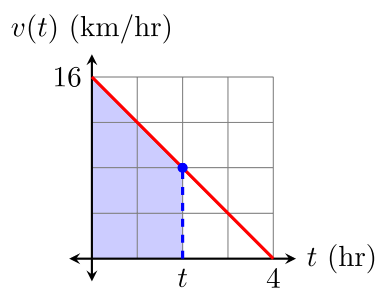
2. This problem is similar to Example 2.



Since we are at 60 km/hr after 3 hr, the slope of the line must be $\frac{60}{3} = 20$. Since the line goes through the origin, the equation of the line is $v(t) = 20t$. Finding the area of the triangle shown above (just as in Example 2), we get

$$\begin{aligned} s(t) &= \frac{1}{2} \cdot b \cdot h \\ &= \frac{1}{2} \cdot t \cdot 20t \\ &= 10t^2. \end{aligned}$$

3. This problem is similar to Example 3.



To find the area of the trapezoid, we first need to find an equation for the line. We see that the line passes through $(0, 16)$ and $(4, 0)$. Thus, the slope is

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{0 - 16}{4 - 0} \\ &= -4. \end{aligned}$$

Remember that b is the y-intercept, which is 16 in this example. So an equation for the line is $v(t) = -4t + 16$.

Here, b_1 is always 16, b_2 corresponds to $v(t)$, and h is just t. This means that

$$\begin{aligned} s(t) &= \frac{1}{2} \cdot (b_1 + b_2) \cdot h \\ &= \frac{1}{2} \cdot (16 + v(t)) \cdot t \\ &= \frac{1}{2}(16 + (-4t + 16)) \cdot t \\ &= \frac{1}{2}(32 - 4t) \cdot t \\ &= 16t - 2t^2. \end{aligned}$$

Chapter 2

The First Derivative

2.1 The Derivative

Example 1

Please visit the link desmos page on Secant Lines for an interactive demonstration of the geometry of secant lines. The purpose of this demo is to see how slopes of secant lines approach the slope of the tangent line through a point.

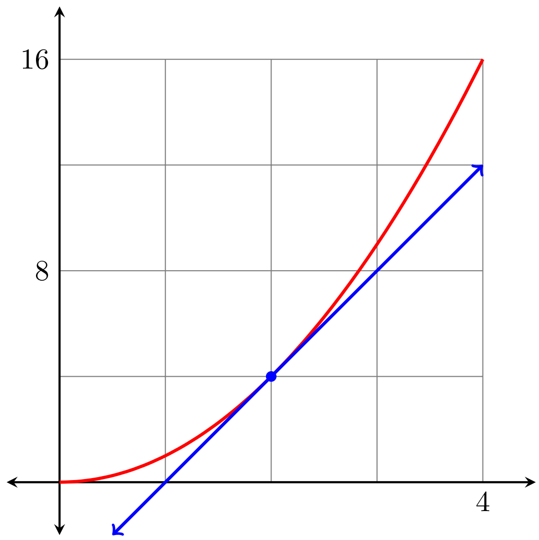


Figure 2.1: Graph of $y = x^2$ on the domain $[0, 4]$.

Why is this important? If you look at Figure 2.1, you can see the tangent line drawn through $(2, 4)$. Now to find the equation of this line, we either need two points, or a point and a slope. All we have is one point.

Using the geometry of secant lines, we will be able to calculate the slope of the tangent line. So let's look at this from an algebraic point of view. In the demonstration, we focused on the value $x = 3$, but now we'll look at the same geometry where x can be any value.

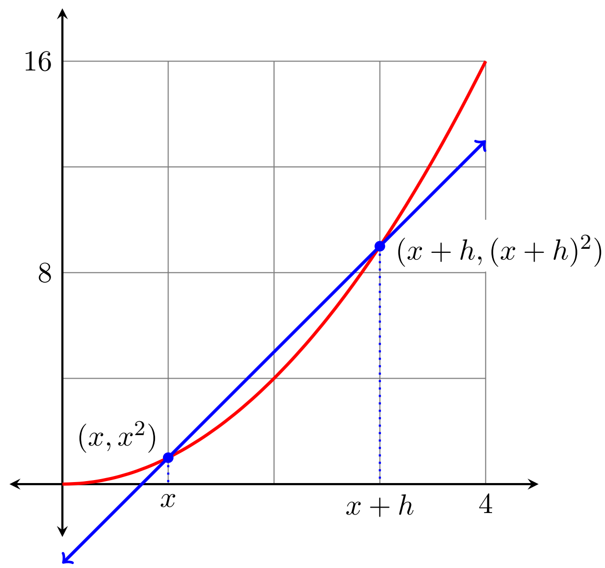


Figure 2.2: Graph of $y = x^2$ on the domain $[0, 4]$.

Begin with the point (x, x^2) on the graph of $y = x^2$. To find a secant line through this point, consider an x-value of $x+h$, with y-value $(x+h)^2$, so the other point on the graph is $(x+h, (x+h)^2)$. Draw the secant line between these two points, as shown in Figure 2.2.

Now use the slope formula to calculate the slope. Let the points be $(x_1, y_1) = (x, x^2)$ and $(x_2, y_2) = (x+h, (x+h)^2)$. Then

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{(x+h)^2 - x^2}{(x+h) - x} \\ &= \frac{(x+h)^2 - x^2}{h} \end{aligned}$$

Remember, we want to let h get smaller and smaller until it's eventually 0. The problem is that if we try to substitute $h = 0$ at this point, we'll get $\frac{0}{0}$, which is undefined.

To solve this problem, we use the idea of a **limit** in calculus. The idea of a limit is *the fundamental* new concept in calculus which you likely haven't seen in precalculus. We're thinking along these lines: "If we look at the slopes of secant lines for small values of h, it looks like they're all approaching the same value. But we can't actually plug in $h = 0$, since we'll get $\frac{0}{0}$. So we need to take the limit of these slopes as h approaches 0."

There is a new calculus notation for this. It's

$$m = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}.$$

We read it as "m is the limit as $h \to 0$ of $(x+h)^2 - x^2$ over h."

Think of it this way. As we saw in the demo, the idea of a tangent line is very geometrical. But actually calculating the slopes of tangent lines takes a bit of algebra. The limit is the concept in calculus which brings together the geometrical and algebraic aspects of calculus.

Let's continue working to find the slope. The next step is to simplify our expression for m as much as possible and see where that leaves us.

$$\begin{aligned} m &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h\to 0} (2x+h). \end{aligned}$$

That's as far as we can go. But do you notice what happened? *Now* we can plug in $h = 0$. That's because we canceled out the h from the denominator, so we're not dividing by 0 any more.

So we have

$$\begin{aligned} m &= \lim_{h\to 0} (2x+h) \\ &= 2x + 0 \\ &= 2x. \end{aligned}$$

But exactly what does the "$2x$" tell us? Look back at Figure 2.1. What is the equation of the tangent line at $x = 2$, which is the line shown there? Since the slope of a tangent line is $2x$, then the slope of this line has to be $2 \cdot 2 = 4$. This means that we are looking for the line passing through $(2, 4)$ with a slope of 4. Using the method of your choice, you get $y = 4x - 4$.

What have we done? Recall that the problem with finding an equation for a tangent line is that a tangent line is defined to be a line touching a curve at *one* point. We do not have a second point. But by using the concept of a *limit,* we can find the slope of the tangent line by taking a limit of the slopes of secant lines.

Let's now look at some common calculus notation. While we wrote $y = x^2$, very often we write $f(x) = x^2$. Depending on which notation is used (both are common), we would write $\frac{dy}{dx} = 2x$ or $f'(x) = 2x$, and call $2x$ the **derivative** of y, or the **derivative** of $f(x)$.

Rewriting our previous work using $f'(x)$, we would say that

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}.$$

But $f(x) = x^2$ and $f(x+h) = (x+h)^2$, so we could also say that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Written this way, we say that given the function $f(x)$, this is the **definition of the derivative** of $f(x)$. Deserves a double box.

Definition of the Derivative

If $f(x)$ is a function, we define the **derivative** of $f(x)$ to be $f'(x)$, which is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Example 2

Find the derivative of $f(x) = \sqrt{x}$.

Let's use our new definition to do this. We have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}. \end{aligned}$$

How can we simplify this? The trick, which we saw earlier, is to rationalize the numerator. Let's see what happens.

$$\begin{aligned} \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h\to 0} \frac{x + h + \sqrt{x+h}\cdot\sqrt{x} - \sqrt{x}\cdot\sqrt{x+h} - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}. \end{aligned}$$

Stop and look. What just happened here is because we rationalized the numerator, we got to a point where we could cancel the h in the denominator. And remember, since we want h to go to 0, we *have* to be able to cancel the h if we want to go any further.

Because when we cancel that h, *now* we can plug in $h = 0$.

$$\begin{aligned} \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} &= \frac{1}{\sqrt{x+0} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

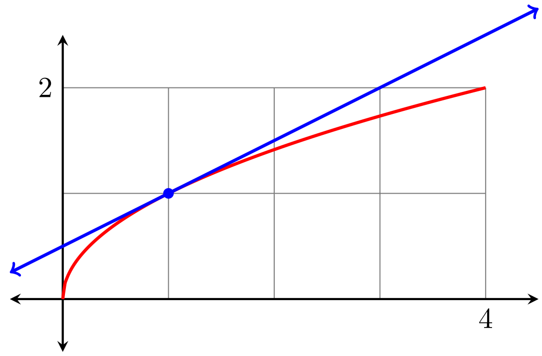


Figure 2.3: Graph of $y = \sqrt{x}$ on the domain $[0, 4]$.

Let's see how we would use the fact that $f'(x) = \dfrac{1}{2\sqrt{x}}$. A graph of $f(x)$ is shown in Figure 6.19, with a tangent line drawn at the point $(1, 1)$.

How would we get an equation of the tangent line? The slope is given by

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2},$$

and a point on the tangent line is $(1, 1)$. So an equation for the tangent line is $y = \dfrac{1}{2}x + \dfrac{1}{2}$.

Example 3

What is $f'(x)$ if $f(x) = \dfrac{1}{x}$? Again, we go back and use the definition of the derivative. Note that we multiply both top and bottom by $(x + h)x$ since that is the common denominator of the fractions in the numerator.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} \\
&= \lim_{h \to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} \cdot \frac{(x+h)x}{(x+h)x} \\
&= \lim_{h \to 0} \frac{\dfrac{1}{x+h} \cdot (x+h)x - \dfrac{1}{x} \cdot (x+h)x}{h(x+h)x} \\
&= \lim_{h \to 0} \frac{x - (x+h)}{h(x+h)x} \\
&= \lim_{h \to 0} \frac{x - x - h}{h(x+h)x} \\
&= \lim_{h \to 0} \frac{-h}{h(x+h)x}
\end{aligned}$$

Now we can cancel out the h. When using the definition of the derivative, the h must *always* cancel out. *Always.* This is one way to know that you're on the right track. So

$$f'(x) = \lim_{h \to 0} \frac{-1}{(x+h)x}.$$

Because the h canceled, plugging in $h = 0$ is not a problem anymore. So

$$\begin{aligned} \lim_{h \to 0} \frac{-1}{(x+h)x} &= \frac{-1}{(x+0)x} \\ &= -\frac{1}{x^2}. \end{aligned}$$

One thing you probably noticed is that these problems have involved a *lot* of algebra. There is no way around this – you have to keep simplifying until you can get the h to cancel. Once you *do* get the h to cancel, you can be pretty sure you're on the right track.

Example 4

As mentioned, there is usually quite a bit of algebra in working with the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

So let's break it down into steps.

Let's start with $f(x) = x^2 - 3x$. How do we evaluate $f(x+h)$? We substitute $x + h$ everywhere we see an x. If you sometimes get stuck with this, here's something to try.

1. First, rewrite the function with boxes.

$$f\left(\boxed{}\right) = \left(\boxed{}\right)^2 - 3\left(\boxed{}\right).$$

2. Next, put $x + h$ in each empty box.

$$f\left(\boxed{x+h}\right) = \left(\boxed{x+h}\right)^2 - 3\left(\boxed{x+h}\right).$$

3. We don't need the boxes any more.

$$f(x+h) = (x+h)^2 - 3(x+h).$$

4. Expand. Be careful when distributing the minus sign.

$$f(x+h) = x^2 + 2xh + h^2 - 3x - 3h.$$

5. Now substitute into the limit definition and simplify until the h cancels. Again, watch the minus signs. Note that for the h to cancel, *every* term in the numerator that does *not* contain h should cancel.

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{x^2+2xh+h^2-3x-3h-(x^2-3x)}{h}\\
&= \lim_{h\to 0}\frac{x^2+2xh+h^2-3x-3h-x^2+3x}{h}\\
&= \lim_{h\to 0}\frac{2xh+h^2-3h}{h}\\
&= \lim_{h\to 0}\frac{h(2x+h-3)}{h}\\
&= \lim_{h\to 0}(2x+h-3)\\
&= 2x-3.
\end{aligned}$$

Assessment Expectations: You do *not* have to find the derivative using boxes. But I have found that many students start off with an incorrect expression for $f(x+h)$. If your first step isn't correct, it's going to be difficult to get the h to cancel. Use it only if it helps you.

Homework

1. Using the definition of the derivative, find $f'(x)$ if $f(x) = -3x$.

2. (a) Using the definition of the derivative, find $f'(x)$ if $f(x) = x - 2x^2$.

   (b) Find the equation of the tangent line at $x = 1$. Graph both $f(x)$ and the tangent line on desmos to visually verify that you have the correct tangent line.

3. (a) Using the definition of the derivative, find $f'(x)$ if $f(x) = \sqrt{x+2}$.

   (b) Find the equation of the tangent line at $x = 2$. Graph both $f(x)$ and the tangent line on desmos to visually verify that you have the correct tangent line.

4. (a) Using the definition of the derivative, find $f'(x)$ if $f(x) = \dfrac{1}{x-1}$.

   (b) Find the equation of the tangent line at $x = 3$. Graph both $f(x)$ and the tangent line on desmos to visually verify that you have the correct tangent line.

Solutions

1.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{-3(x+h)-(-3x)}{h} \\ &= \lim_{h\to 0} \frac{-3x-3h+3x}{h} \\ &= \lim_{h\to 0} \frac{-3h}{h} \\ &= \lim_{h\to 0}(-3) \\ &= -3. \end{aligned}$$

2. (a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{x+h-2(x+h)^2-(x-2x^2)}{h} \\ &= \lim_{h\to 0} \frac{x+h-2x^2-4xh-2h^2-x+2x^2}{h} \\ &= \lim_{h\to 0} \frac{h-4xh-2h^2}{h} \\ &= \lim_{h\to 0} \frac{h(1-4x-2h)}{h} \\ &= \lim_{h\to 0}(1-4x-2h) \\ &= 1-4x-2(0) \\ &= 1-4x. \end{aligned}$$

(b) $f(1) = -1$ and $f'(1) = 1-4(1) = -3$, so the slope of the line is -3 and the line passes through $(1,-1)$. This results in the line $y = -3x+2$.

3. (a)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+2+h}-\sqrt{x+2}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+2}-\sqrt{x+2}}{h} \cdot \frac{\sqrt{x+h+2}+\sqrt{x+2}}{\sqrt{x+h+2}+\sqrt{x+2}} \\
&= \lim_{h\to 0} \frac{x+h+2+\sqrt{x+h+2}\cdot\sqrt{x+2}-\sqrt{x+2}\cdot\sqrt{x+h+2}-x-2}{h(\sqrt{x+h+2}+\sqrt{x+2})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+2}+\sqrt{x+2})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h+2}+\sqrt{x+2}} \\
&= \frac{1}{\sqrt{x+0+2}+\sqrt{x+2}} \\
&= \frac{1}{\sqrt{x+2}+\sqrt{x+2}} \\
&= \frac{1}{2\sqrt{x+2}}
\end{aligned}$$

(b) $f(2) = 2$ and $f'(2) = \dfrac{1}{2\sqrt{2+2}} = \dfrac{1}{4}$, and so we are looking for a line with slope $\dfrac{1}{4}$ which passes through the point $(2, 2)$. This results in $y = \dfrac{1}{4}x + \dfrac{3}{2}$.

4. (a)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\dfrac{1}{x+h-1} - \dfrac{1}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{x+h-1} - \dfrac{1}{x-1}}{h} \cdot \frac{(x+h-1)(x-1)}{(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{x+h-1} \cdot (x+h-1)(x-1) - \dfrac{1}{(x-1)} \cdot (x+h-1)(x-1)}{h(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{x-1-(x+h-1)}{h(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{x-1-x-h+1}{h(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{-h}{h(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} \\
&= \frac{-1}{(x+0-1)(x-1)} \\
&= -\frac{1}{(x-1)^2}
\end{aligned}$$

(b) $f(3) = \dfrac{1}{2}$ and $f'(3) = -\dfrac{1}{(3-1)^2} = -\dfrac{1}{4}$, so we are looking for a line with slope $-\dfrac{1}{4}$ which passes through the point $\left(3, \dfrac{1}{2}\right)$. This line is $y = -\dfrac{1}{4}x + \dfrac{5}{4}$.

2.2 The Derivative of $y = \sin(x)$.

Let's look at the derivatives of a few common functions. We'll start with $f(x) = \sin(x)$; two full periods are graphed in Figure 2.4. What is the slope of the tangent line at $x = 0$? In other words, what is $f'(0)$?

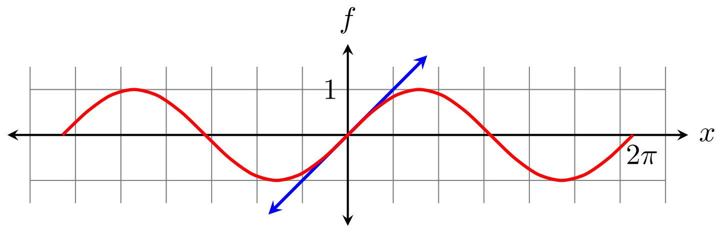


Figure 2.4: Graph of $f(x) = \sin(x)$ with tangent line at $x = 0$.

We will use the following definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

We first start with $x = 0$, write out the definition substituting in 0 for x, and then simplify a little bit.

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{\sin(h) - \sin(0)}{h} && \text{since } f(x) = \sin(x) \\ &= \lim_{h\to 0} \frac{\sin(h)}{h} && \text{since } \sin(0) = 0. \end{aligned}$$

Up until now, we were able to use algebra to make the "h" cancel out so we could just substitute $h = 0$. But it is not possible to do that here. So how do we proceed?

There are two other ways we can look at limits: numerically and graphically. We'll start with numerically. Since we are looking at a limit at $h \to 0$, you can use your calculator to look at the quotient $\dfrac{\sin(h)}{h}$ for values of h closer and closer to 0.

I set my calculator to radian mode (important!) and rounded to six decimal places. As h gets closer to 0 from the left and right, it looks like the quotient $\dfrac{\sin(h)}{h}$ gets closer and closer to 1. Using limit notation, we would write

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1.$$

h	$\sin(h)/h$
-0.1	0.998334
-0.01	0.999983
-0.001	1.000000
-0.0001	1.000000
0.1	0.998334
0.01	0.999983
0.001	1.000000
0.0001	1.000000

Table 2.1: Approximating $\lim\limits_{h\to 0} \dfrac{\sin(h)}{h}$.

It is worth noting that if your calculator were in degree mode, it would look like this limit is approximately 0.017453. Units of radians make trigonometry much easier (as far as calculus is concerned). This is very similar to choosing appropriate units in science. The metric system is far better suited to science than inches, ounces, etc.

Another way to guess $\lim\limits_{h\to 0} \dfrac{\sin(h)}{h}$ is to look at the ratio $\dfrac{\sin(h)}{h}$ as a function itself, as in Figure 2.5.

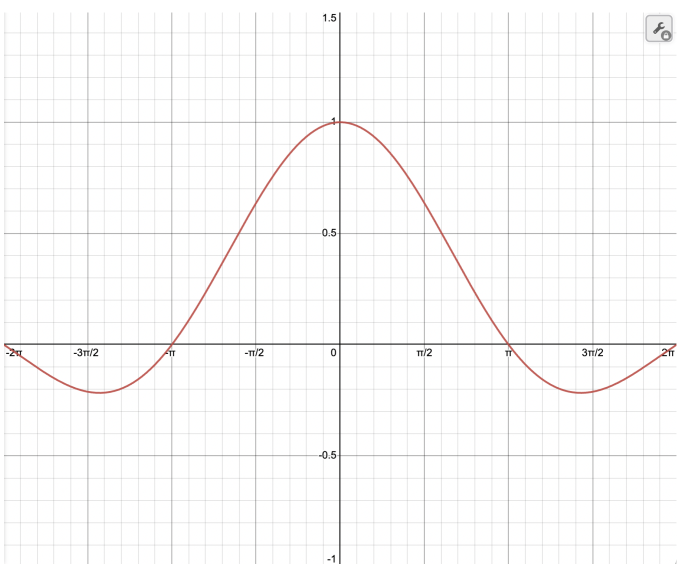


Figure 2.5: Graph of $g(h) = \dfrac{\sin(h)}{h}$.

I made this graph with desmos and took a screen shot. It looks like it crosses the y-axis at 1 (just like the limit). It's important to say "looks like" since you can't actually evaluate

$$g(0) = \frac{\sin(0)}{0} = \frac{0}{0}.$$

But most graphing programs are able to "fill in the hole" at $x = 0$ to get a smooth curve.

There is also a more complicated mathematical proof using geometry and trigonometry, but it's more than we need. For the functions we'll be looking at, if looking at a limit numerically and graphically gives the same result, then you can be sure you've found the right limit.

To recap, we found that

$$\boxed{\lim_{h \to 0} \frac{\sin(h)}{h} = 1.}$$

But this was the limit we needed to evaluate to find the slope of the tangent line at $x = 0$, and so the slope of this line is 1.

We started just with looking at $x = 0$ since we needed to see different ways to evaluate limits. So with $f(x) = \sin(x)$, let's find $f'(x)$ for *every* x. To do this, we'll need an identity from trigonometry:

$$\boxed{\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b).}$$

Now let's start with the definition.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} && \text{since } f(x) = \sin(x) \\
&= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} && \text{using the identity}
\end{aligned}$$

This looks a bit more complicated than other limits we've seen. Let's take a few steps to rearrange terms.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \lim_{h \to 0} \left[\sin(x) \cdot \frac{\cos(h)}{h} + \cos(x) \cdot \frac{\sin(h)}{h} - \frac{\sin(x)}{h}\right] && \text{splitting apart} \\
&= \lim_{h \to 0} \left[\sin(x)\left(\frac{\cos(h) - 1}{h}\right) + \cos(x) \cdot \frac{\sin(h)}{h}\right] && \text{combining } \sin(x) \text{ terms}
\end{aligned}$$

Now since h goes to 0, x does not change in this limit. So we can factor out terms *only* containing x. If the limit involves $h \to a$, you can *never* factor out any expression containing h from the limit.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left[\sin(x)\left(\frac{\cos(h)-1}{h}\right)+\cos(x)\cdot\frac{\sin(h)}{h}\right] \\ &= \sin(x)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} \end{aligned}$$

The first limit might not look familiar, but the second one does – we started off by finding this exact limit: it is 1.

What about the first limit? Since we already worked one limit like this in detail, we won't do another one. But when you look at this limit numerically and graphically, you see that:

$$\boxed{\lim_{h\to 0}\frac{\cos(h)-1}{h}=0.}$$

How can we use this? Basically, we rewrote the quotient $\dfrac{f(x+h)-f(x)}{h}$ in such a way that is involves limits we can derive numerically and graphically. So we just substitute in the values of these limits.

$$\begin{aligned} f'(x) &= \sin(x)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} \\ &= \sin(x)\cdot 0+\cos(x)\cdot 1 \\ &= \cos(x). \end{aligned}$$

Done! So when $f(x)=\sin(x)$, then $f'(x)=\cos(x)$. So $\cos(x)$ is the derivative of $\sin(x)$, which we often write

$$\boxed{\frac{d}{dx}\sin(x)=\cos(x).}$$

Exercises

1. Show numerically and graphically that

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0.$$

2. By following a similar sequence of steps as for $\sin(x)$, but using a different trigonometric identity, show that

$$\frac{d}{dx} \cos(x) = -\sin(x).$$

3. Let $f(x) = \sqrt{x+1}$, so that $f'(x) = \dfrac{1}{2\sqrt{x+1}}$.

   (a) Graph this function on desmos.

   (b) Where is $f'(x)$ defined?

   (c) What is $f'(3)$?

   (d) Find the equation of the tangent line at $x = 3$.

4. Let $f(x) = x^3 - 3x$.

   (a) Graph this function on desmos.

   (b) Using the definition of the derivative, show that $f'(x) = 3x^2 - 3$.

   (c) Where is $f'(x)$ defined?

   (d) What is $f'(2)$?

   (e) Find the equation of the tangent line at $x = 2$.

Solutions

1. Create a chart like we did for $\dfrac{\sin(h)}{h}$ and observe the numbers keep getting closer to 0 as h gets closer to 0. Note: if your numbers are not matching, be sure your calculator is in radian mode.

h	$(\cos(h)-1)/h$
-0.1	0.049958
-0.01	0.005000
-0.001	0.000500
-0.0001	0.000050
0.1	0.049958
0.01	0.005000
0.001	0.000500
0.0001	0.000050

You can check this graphically using desmos or a graphing calculator.

2. The identity we need is

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

Now let's use the definition of the derivative with $f(x) = \cos(x)$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h} && \text{since } f(x)=\cos(x) \\ &= \lim_{h\to 0} \frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h} && \text{using the identity} \end{aligned}$$

Now rearrange terms.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h} \\ &= \lim_{h\to 0} \left[\cos(x)\cdot\frac{\cos(h)}{h} - \sin(x)\cdot\frac{\sin(h)}{h} - \cos(x)\cdot\frac{1}{h}\right] && \text{splitting apart} \\ &= \lim_{h\to 0} \left[\cos(x)\left(\frac{\cos(h)-1}{h}\right) - \sin(x)\cdot\frac{\sin(h)}{h}\right] && \text{combining } \cos(x) \text{ terms} \end{aligned}$$

Recall the following limits.

$$\lim_{h\to 0}\frac{\cos(h)-1}{h} = 0, \qquad \lim_{h\to 0}\frac{\sin(h)}{h} = 1.$$

Substituting in, we get

$$\begin{aligned} f'(x) &= \cos(x) \cdot 0 - \sin(x) \cdot 1 \\ &= -\sin(x) \end{aligned}$$

3. (a) Do this online.
   (b) $f'(x)$ is defined when $x+1>0$, since you can't have 0 in the denominator or a negative number inside a square root. So $x > -1$, or $(-1, \infty)$ using interval notation.
   (c)
$$f'(3) = \frac{1}{2\sqrt{3+1}} = \frac{1}{4}.$$
   (d) We know the slope is $\frac{1}{4}$ from (c). $f(3) = \sqrt{3+1} = 2$, so a point on the line is $(3, 2)$.
$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= \frac{1}{4}(x-3) \\ y - 2 &= \frac{1}{4}x - \frac{3}{4} \\ y &= \frac{1}{4}x + \frac{5}{4} \end{aligned}$$

   (a) Do this online.
   (b)
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - 3(x+h) - (x^3 - 3x)}{h} \\ &= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{3x} - 3h - \cancel{x^3} + \cancel{3x}}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} \\ &= \lim_{h\to 0} \left(\frac{3x^2h}{h} + \frac{3xh^2}{h} + \frac{h^3}{h} - \frac{3h}{h}\right) \\ &= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 3) \\ &= 3x^2 - 3. \end{aligned}$$
   (c) Polynomials are defined for *all* x. In interval notation, $f(x)$ is defined on $(-\infty, \infty)$.
   (d)
$$f'(2) = 3(2^2) - 3 = 9.$$
   (e) We know the slope is 9 from (d). Since $f(2) = 2$, we use the point $(2, 2)$.
$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 9(x-2) \\ y - 2 &= 9x - 18 \\ y &= 9x - 16 \end{aligned}$$

2.3 The Geometry of Derivatives

We just learned how to find a derivative using the geometric definition derived from looking at secant lines. The process of finding a derivative *algebraically* is sometimes rather tedious. Here, we'll look at the *geometrical* meaning of the derivative. Because we want to emphasize the important concepts, we'll look at a basic function, $f(x) = x^2$, shown below, with it's derivative, $f'(x) = 2x$.

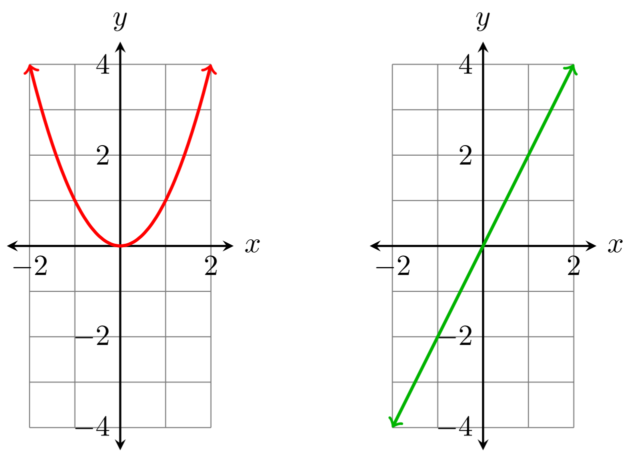


Figure 2.6: Graph of $f(x) = x^2$ (left) with derivative $f'(x) = 2x$ (right).

Let's look at exactly what knowledge we can gain by knowing the derivative. First, we can find the slope of a tangent at any given point. So, since $f(1) = 1$, we know that the tangent line goes through $(1, 1)$. And since $f'(x) = 2x$, this line has a slope of $f'(1) = 2 \cdot 1 = 2$, shown below. Note the corresponding point on the derivative graph, $(1, 2)$.

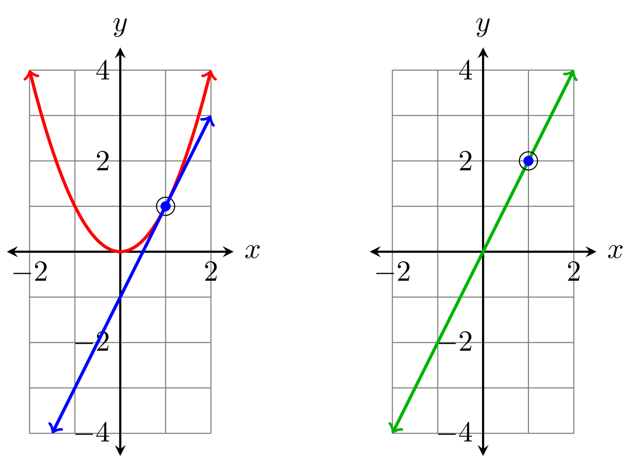


Figure 2.7: Graph of $f(x) = x^2$ (left) with derivative $f'(x) = 2x$ (right).

We now have enough information to work out an equation for the tangent line at $x = 1$, since we know the slope $m = 2$ and a point $(1, 1)$ on the line. For reference, we recall that a line with slope m which passes through the point (x_1, y_1) can be described by the following equation:

$$y - y_1 = m(x - x_1).$$

Substituting in our values: $m = 2$, $x_1 = 1$, and $y_1 = 1$, we get

$$y - 1 = 2(x - 1),$$

which simplifies to $y = 2x - 1$. In Figure 2.7, you can observe that the slope is 2 and the y-intercept is -1. So we can use the derivative to find an equation of the tangent line at a specific point.

What else does the derivative tell us? Let's look now at the case when $x > 0$; we look at the specific case $x = 0.5$ in the graphs below.

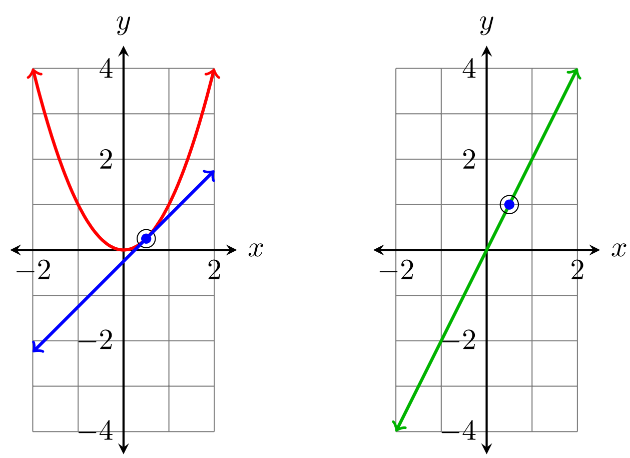


Figure 2.8: Graph of $f(x) = x^2$ (left) with derivative $f'(x) = 2x$ (right); $x = 0.5$.

Notice that the graph is *increasing* when $x > 0$, and so the tangent line has a *positive* slope. We can see this by looking at the graph. But in addition to this, we have, when $x > 0$,

$$\begin{aligned} f'(x) &= 2x \\ &> 2 \cdot 0 && \text{since } x > 0 \\ &= 0. \end{aligned}$$

We can summarize this as follows:

If $f'(x) > 0$ for some value of x, then the function $f(x)$ is increasing at that value of x.

Now let's look at the case when $x < 0$; the case when $x = -1.5$ is graphed in Figure 2.9.

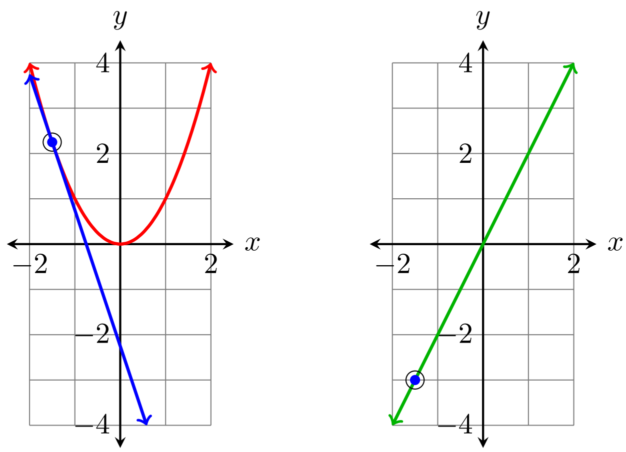


Figure 2.9: Graph of $f(x) = x^2$ (left) with derivative $f'(x) = 2x$ (right); $x = -1.5$.

In this case, the graph is *decreasing* when $x < 0$, and so the tangent line has a *negative* slope. We can see this by looking at the graph. But we can also see this algebraically; when $x < 0$,

$$\begin{aligned} f'(x) &= 2x \\ &< 2 \cdot 0 && \text{since } x < 0 \\ &= 0. \end{aligned}$$

We can summarize this case as follows:

> If $f'(x) < 0$ for some value of x, then the function $f(x)$ is decreasing at that value of x.

Let's summarize what we've observed so far.

$f'(x)$	Where	What happens
$f'(x) < 0$	$(-\infty, 0)$	$f(x)$ is decreasing
$f'(x) > 0$	$(0, \infty)$	$f(x)$ is increasing

So we can understand some features of the graph of a function by looking at its derivative. It is *always* true that if $f'(x) < 0$, then $f(x)$ is decreasing, and if $f'(x) > 0$, then $f(x)$ is increasing. But the case when $f'(x) = 0$ is a little trickier. We'll be looking at this case in detail later.

Exercises

1. Using the definition of the derivative, find $f'(x)$ if $f(x) = 3$. Write a short sentence interpreting this geometrically.

2. Using the definition of the derivative, show that if $f(x) = ax$, then $f'(x) = a$. In this example, a is just a number, like 3. Write a short sentence interpreting this geometrically.

3. Let $f(x) = \cos(x)$.

   (a) Graph this function on desmos. Use a domain of $[0, 2\pi]$ (you can use the wrench icon in the upper right and type "pi" for π).

   (b) What is $f'\left(\frac{\pi}{2}\right)$?

   (c) Find the equation of the tangent line at $x = \frac{\pi}{2}$.

   (d) Where is $f'(x) = 0$? Remember, the domain is $[0, 2\pi]$. By inspecting the graph, decide if there is a minimum, maximum, or inflection point at these values.

   (e) Where is the function increasing?

   (f) Where is the function decreasing?

4. Let $f(x)$ be a function – you don't know exactly what $f(x)$ is, but you are given that $f'(x) = x^2(x-2)^2$. The function is defined on all real numbers.

   (a) Where is this function increasing?

   (b) Where is this function decreasing?

   (c) When is $f'(x) = 0$? Based on what you found in (a) and (b), decide if $f(x)$ has a minimum, maximum, or inflection point at these values.

   (d) You are given that $f(3) = 12$. Find an equation of the tangent line at $x = 3$.

Solutions

1.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{3-3}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0 \end{aligned}$$

Since $y = 3$ is a horizontal line, this means that its slope is 0.

2.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{a(x+h)-ax}{h} \\ &= \lim_{h\to 0} \frac{ax+ah-ax}{h} \\ &= \lim_{h\to 0} \frac{ah}{h} \\ &= \lim_{h\to 0} a \\ &= a. \end{aligned}$$

This confirms that the line $y = ax$ is a line with slope a.

3. (a) Do this online.

(b) $f'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1.$

(c) We know the slope is -1 from (b). Since $f\left(\frac{\pi}{2}\right) = 0$, we use the point $\left(\frac{\pi}{2}, 0\right)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= -1\left(x - \frac{\pi}{2}\right) \\ y &= -x + \frac{\pi}{2} \end{aligned}$$

(d) $f'(x) = -\sin(x) = 0$ exactly when $x = 0, \pi, 2\pi$ (given our knowledge of the unit circle and the fact that the domain is $[0, 2\pi]$). Looking at the graph, we see a local maximum at $x = 0$ and $x = 2\pi$, and a local minimum at $x = \pi$.

(e) The function is increasing wherever we have $f'(x) > 0$. Looking at the graph of $y = -\sin(x)$ on desmos, we observe that $f'(x) = -\sin(x)$ is positive on the interval $(\pi, 2\pi)$. By visually inspecting the graph of $f(x) = \cos(x)$, we observe that $f(x)$ is increasing on this interval.

(f) The function is decreasing wherever we have $f'(x) < 0$. Looking at the graph of $y = -\sin(x)$ on desmos, we observe that $f'(x) = -\sin(x)$ is negative on the interval $(0, \pi)$. By visually inspecting the graph of $f(x) = \cos(x)$, we observe that $f(x)$ is decreasing on this interval.

4. In this problem, you are not given the graph of the function, but you should still be able to answer the following questions.

(a) $f'(x) = x^2(x-2)^2$, but x^2 and $(x-2)^2$ are both positive. So $f'(x)$ is always positive, therefore $f(x)$ is always increasing. In interval notation, this would be $(-\infty, \infty)$.

(b) Based on the answer to (a), $f(x)$ is never decreasing.

(c) Since $f'(x)$ is in factored form, the zeros are easy to find: $x = 0$ and $x = 2$. Now if there were a minimum at $x = 0$, we would go from decreasing to increasing, which is impossible since $f(x)$ is never decreasing. Likewise, if there were a maximum, we would go from increasing to decreasing, again impossible. So there must be inflection points at these two values of x.

(d) Since $f'(3) = 3^2 \cdot (3-2)^2 = 9$, the slope of the tangent line is 9. Since $f(3) = 12$, we know that $(3, 12)$ is a point on the tangent line. We can use these to get an equation of the line.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 12 &= 9(x - 3) \\ y - 12 &= 9x - 27 \\ y &= 9x - 15 \end{aligned}$$

Chapter 3

Using Rules of Differentiation

3.1 Rules of Differentiation

We're finished using the definition of the derivative for a while. Now we want to learn how to use some rules which will make differentiating more complex functions possible.

Example 1

We have seen that $\frac{d}{dx}x = 1$, $\frac{d}{dx}x^2 = 2x$. Using the definition of the derivative, we see that

$$\begin{aligned}
\frac{d}{dx}x^3 &= \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0}\frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h} \\
&= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0}\left(\frac{3x^2h}{h} + \frac{3xh^2}{h} + \frac{h^3}{h}\right) \\
&= \lim_{h\to 0}(3x^2 + 3xh + h^2) \\
&= 3x^2.
\end{aligned}$$

Note that dividing through by h is an alternative to factoring the h out. Both methods will work.

This pattern continues, and we have the Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

This works even if the exponent is not a positive integer. We've seen this twice before. Using the Power Rule, we have

$$\begin{aligned}
\frac{d}{dx}\frac{1}{x} &= \frac{d}{dx}x^{-1} \\
&= -1x^{-1-1} \\
&= -x^{-2} \\
&= -\frac{1}{x^2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dx}\sqrt{x} &= \frac{d}{dx}x^{1/2} \\
&= \frac{1}{2}x^{1/2-1} \\
&= \frac{1}{2}x^{-1/2} \\
&= \frac{1}{2\sqrt{x}},
\end{aligned}$$

both found before using the definition of the derivative. Now we have the Power Rule, and so can use this instead of the definition.

Example 2

We have seen that $\frac{d}{dx}x^2 = 2x$ and $\frac{d}{dx}\sin(x) = \cos(x)$. Adding or subtracting functions, or multiplying functions by a number, do *not* affect the algebra needed in the definition of a derivative. So

$$\begin{aligned}\frac{d}{dx}(x^2 + \sin(x)) &= 2x + \cos(x)\\ \frac{d}{dx}(\sin(x) - x^2) &= \cos(x) - 2x\\ \frac{d}{dx}5x^2 &= 5(2x) = 10x\\ \frac{d}{dx}(-3\sin(x)) &= -3\cos(x)\end{aligned}$$

Example 3

However, multiplying and dividing functions *does* affect the algebra when using the definition of the derivative. So you cannot just multiply or divide derivatives. Let's look at multiplying functions first. Here, we use the Product Rule:

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$$

We won't prove this here, but let's look at an example to see how to use it. We'll find $\frac{d}{dx}x\cos(x)$. Here, $f(x) = x$ and $g(x) = \cos(x)$, so that

$$\begin{aligned}f(x) &= x & f'(x) &= 1\\ g(x) &= \cos(x) & g'(x) &= -\sin(x)\end{aligned}$$

Now substitute in the Product Rule.

$$\begin{aligned}\frac{d}{dx}x\cos(x) &= f(x)g'(x) + g(x)f'(x)\\ &= x(-\sin(x)) + \cos(x)\cdot 1\\ &= -x\sin(x) + \cos(x)\end{aligned}$$

This is much easier than going back and using the definition of the derivative.

Example 4

When dividing functions, we use the Quotient Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

This looks a bit more complicated, but as long as you substitute carefully, you'll be fine. We'll use it to find $\frac{d}{dx}\frac{\sin(x)}{x^2}$. Here are the substitutions:

$$f(x) = \sin(x) \qquad f'(x) = \cos(x)$$
$$g(x) = x^2 \qquad g'(x) = 2x$$

Now substitute into the Quotient Rule:

$$\begin{aligned}\frac{d}{dx}\frac{\sin(x)}{x^2} &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}\\ &= \frac{x^2 \cdot \cos(x) - \sin(x)(2x)}{(x^2)^2}\\ &= \frac{x^2\cos(x) - 2x\sin(x)}{x^4}\\ &= \frac{x(x\cos(x) - 2\sin(x))}{x^4}\\ &= \frac{x\cos(x) - 2\sin(x)}{x^3}\end{aligned}$$

Notice that the x cancels. Be sure to make any simple cancellations when possible.

Example 5

Let's briefly review function composition. Suppose $f(x) = \cos(x)$ and $g(x) = x^3$. Then

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= \cos(g(x))\\ &= \cos(x^3).\end{aligned}$$

For taking derivatives using the Chain Rule (done in the next section), we'll need to do this in *reverse.* As an example, if $h(x) = (3x + 2)^4$, find $f(x)$ and $g(x)$ such $h(x) = f(g(x))$.

To think about this, notice that $g(x)$ is the function you evaluate *first,* and $f(x)$ is the function you evaluate *last.* Think about how you would evaluate $h(x)$ using your calculator. If you have to find $h(x)$, the first thing you'd do is evaluate $3 \cdot 5 + 2 = 17$, and the last thing you'd do is take 17^4. So $g(x) = 3x + 2$ and $f(x) = x^4$.

Summary of Rules of Differentiation

Power Rule:	$\frac{d}{dx}(x^n) = nx^{n-1}$
Sum Rule:	$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
Difference Rule:	$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$
Constant Multiple Rule:	$\frac{d}{dx}(cf(x)) = cf'(x)$
Product Rule:	$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$
Quotient Rule:	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$
Chain Rule:	$\frac{d}{dx}(f \circ g)(x) = f'(g(x))g'(x)$

Exercises

1. Find the deriviatives of the following functions using the appropriate rule.
   (a) $h(x) = 3x^5 - x^3$
   (b) $h(x) = 2\cos(x) - x^2$
   (c) $h(x) = x^3 \sin(x)$
   (d) $h(x) = \dfrac{x^2}{\cos(x)}$
2. Review Paul's Online Notes on function composition, if necessary. For each of the following functions, find $f(x)$ and $g(x)$ so that $h = f \circ g$.
   (a) $h(x) = (2x-1)^5$
   (b) $h(x) = \sin^2(x)$
   (c) $h(x) = \dfrac{1}{x^3 + x}$
   (d) $h(x) = \cos(2x+1)$

Solutions

1. (a)

$$\begin{aligned} h'(x) &= \frac{d}{dx}(3x^5 - x^3) \\ &= 3(5x^4) - 3x^2 \\ &= 15x^4 - 3x^2 \end{aligned}$$

(b)

$$\begin{aligned} h'(x) &= \frac{d}{dx}(2\cos(x) - x^2) \\ &= 2(-\sin(x)) - 2x \\ &= -2\sin(x) - 2x \end{aligned}$$

(c) Use the following substitutions in the Product Rule.

$$f(x) = x^3 \qquad f'(x) = 3x^2$$
$$g(x) = \sin(x) \qquad g'(x) = \cos(x)$$

Then

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= x^3\cos(x) + \sin(x)\cdot 3x^2 \\ &= x^3\cos(x) + 3x^2\sin(x). \end{aligned}$$

(d) Use the following substitutions in the Quotient Rule.

$$f(x) = x^2 \qquad f'(x) = 2x$$
$$g(x) = \cos(x) \qquad g'(x) = -\sin(x)$$

Then

$$\begin{aligned} h'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{\cos(x)\cdot 2x - x^2(-\sin(x))}{(\cos(x))^2} \\ &= \frac{2x\cos(x) + x^2\sin(x)}{\cos^2(x)} \end{aligned}$$

2. (a) $f(x) = x^5$, $g(x) = 2x - 1$.
   (b) $f(x) = x^2$, $g(x) = \sin(x)$.
   (c) $f(x) = \dfrac{1}{x}$, $g(x) = x^3 + x$.
   (d) $f(x) = \cos(x)$, $g(x) = 2x + 1$.

3.2 Using Differentiation Rules

When you are given a function $f(x)$ and need to take the derivative, which rule(s) should you use? It's not always obvious.

Example 1

Let $f(x) = \dfrac{x^4 - 3x^2 + 5x}{x}$. Since $f(x)$ is a fraction, your first thought might be to use the quotient rule. This isn't wrong, but it's way too much work. In this case, it's easier to divide out the x first.

$$\begin{aligned} f(x) &= \frac{x^4 - 3x^2 + 5x}{x} \\ &= \frac{x^4}{x} - \frac{3x^2}{x} + \frac{5x}{x} \\ &= x^3 - 3x + 5 \end{aligned}$$

Now, it's easy to find $f'(x)$ using the power rule: $f'(x) = 3x^2 - 3$.

Example 2

Let $f(x) = \dfrac{\cos(x)}{x^{-2}}$. Again, it's tempting to use the quotient rule. But recall that

$$\frac{1}{x^{-2}} = x^2.$$

So it's easier to write $f(x) = x^2 \cos(x)$ and use the product rule.

$$\begin{aligned} f(x) &= x^2 \cos(x) \\ f'(x) &= (x^2)(-\sin(x)) + \cos(x)(2x) \\ &= -x^2 \sin(x) + 2x \cos(x). \end{aligned}$$

Example 3

Let $f(x) = (x^2 + 1)(x - 3)$. Yes, you can use the product rule here. But in this case, it's simpler to FOIL out $f(x)$ and then just use the power rule.

$$\begin{aligned} f(x) &= (x^2 + 1)(x - 3) \\ &= x^3 - 3x^2 + x - 3 \\ f'(x) &= 3x^2 - 6x + 1 \end{aligned}$$

The common theme here is that we rewrote each function so that an easier differentiation rule can be used. There's no "magic formula" for how to do this, you just have to practice. But before jumping into a problem, it's always a good idea to take a moment to see if the function can be rewritten to make it easier to differentiate.

Example 4

Usually, when you see a function inside of another function, you need to use the chain rule. Occasionally, there may be a different way. Suppose $h(x) = (x^3 + 1)^2$. You might try the chain rule with $f(x) = x^2$ and $g(x) = x^3 + 1$. Then $f'(x) = 2x$ and $g'(x) = 3x^2$, so that

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= 2(g(x)) \cdot 3x^2 \\ &= 2(x^3 + 1)(3x^2) \\ &= 6x^2(x^3 + 1). \end{aligned}$$

But it also possible to FOIL out $f(x)$ first. Here's what you get.

$$\begin{aligned} h(x) &= (x^3 + 1)^2 \\ &= x^6 + 2x^3 + 1 \\ h'(x) &= 6x^5 + 6x^2 \end{aligned}$$

Since $6x^2(x^3 + 1) = 6x^5 + 6x^2$, both methods give the same answer. One way isn't necessarily easier than the other, so either way you choose to do it is OK.

Example 5

Let $f(x) = \dfrac{5}{x^6}$. This is a fraction, so you might be tempted to use the quotient rule. But it's easier to use rules of exponents to rewrite $f(x) = 5x^{-6}$. You cannot just use the power rule on the denominator; the *entire* function must be of the form ax^n for some n.

$$\begin{aligned} f(x) &= 5x^{-6} \\ f'(x) &= 5 \cdot (-6)x^{-6-1} \\ &= -30x^{-7} \end{aligned}$$

Note that the exponent must be "-7," not "-5," since we have to subtract 1 from the exponent.

Find the derivatives of the following functions. Some problems will be easier if you rewrite them first, so take a moment to look before you leap.

1. $h(x) = \dfrac{2x}{3x^4}$
2. $h(x) = \dfrac{x^2+1}{x}$
3. $h(x) = \dfrac{x}{x^2+1}$
4. $h(x) = (5x-3)^{10}$
5. $h(x) = (5x-3)^{-10}$
6. $h(x) = x^3\sqrt{x}$
7. $h(x) = \sin(x)\sqrt{x}$
8. $h(x) = \dfrac{ax+b}{ax-b}$ Hint: Treat a and b like numbers, so $\dfrac{d}{dx}a = \dfrac{d}{dx}b = 0$.
9. $h(x) = \sin(x)\cos(x)$
10. $h(x) = \sqrt{3x-5}$
11. $h(x) = \cos(x^3)$
12. $h(x) = \cos^3(x)$
13. $h(x) = \tan(x)$ (Hint: Use the quotient rule.)
14. $h(x) = \dfrac{\sin(x)}{x^{-3}}$
15. For each of the following, simplify/rewrite if possible, and state which rule you would use to take the derivative. Do not actually take the derivative. This problem is for helping you to decide which rule to use.
    (a) $\sin(6x^2+1)$
    (b) $\dfrac{3}{x^7}$
    (c) $\dfrac{\cos(x)}{x^{-2}}$
    (d) $x^2\sqrt{x}$
    (e) $\dfrac{x^3-3x^2}{x^2}$
    (f) $\dfrac{x+1}{x-1}$
    (g) $(x^2+1)^8$
    (h) $\dfrac{3}{x^{-4}}$

Solutions

1. First, rewrite.

$$\begin{aligned}\frac{2x}{3x^4} &= \frac{2}{3x^3}\\ &= \frac{2}{3}x^{-3}\end{aligned}$$

Then use the Power Rule.

$$\begin{aligned}\frac{d}{dx}\frac{2}{3}x^{-3} &= \frac{2}{3}\cdot(-3)x^{-3-1}\\ &= -2x^{-4}\\ &= -\frac{2}{x^4}\end{aligned}$$

2. First rewrite.

$$\begin{aligned}\frac{x^2+1}{x} &= \frac{x^2}{x}+\frac{1}{x}\\ &= x+x^{-1}\end{aligned}$$

Then use the Power Rule.

$$\begin{aligned}\frac{d}{dx}(x^1+x^{-1}) &= 1\cdot x^{1-1}-1\cdot x^{-1-1}\\ &= 1-x^{-2}\\ &= 1-\frac{1}{x^2}\end{aligned}$$

3. Here, we need to use the Quotient Rule.

$$\begin{aligned}f(x) &= x & f'(x) &= 1\\ g(x) &= x^2+1 & g'(x) &= 2x\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\frac{x}{x^2+1} &= \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}\\ &= \frac{(x^2+1)\cdot 1 - x\cdot 2x}{(x^2+1)^2}\\ &= \frac{x^2+1-2x^2}{(x^2+1)^2}\\ &= \frac{1-x^2}{(x^2+1)^2}\end{aligned}$$

Note that we do *not* expand the denominator when using the Quotient Rule.

4. Use the Chain Rule with $f(x) = x^{10}$ and $g(x) = 5x - 3$.

$$f(x) = x^{10} \qquad f'(x) = 10x^9$$
$$g(x) = 5x - 3 \qquad g'(x) = 5$$

Then

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= 10(g(x))^9 \cdot 5 \\ &= 50(5x-3)^9 \end{aligned}$$

5. Use the Chain Rule with $f(x) = x^{-10}$ and $g(x) = 5x - 3$.

$$f(x) = x^{-10} \qquad f'(x) = -10x^{-11}$$
$$g(x) = 5x - 3 \qquad g'(x) = 5$$

Then

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= -10(g(x))^{-11} \cdot 5 \\ &= -50(5x-3)^{-11} \\ &= -\frac{50}{(5x-3)^{11}} \end{aligned}$$

Note that some resources may leave the exponent as negative, while others will rewrite with a positive exponent in the denominator.

6. First, combine exponents.

$$\begin{aligned} x^3\sqrt{x} &= x^3x^{1/2} \\ &= x^{7/2} \end{aligned}$$

Then use the Power Rule.

$$\begin{aligned} h'(x) &= \frac{7}{2}x^{7/2-1} \\ &= \frac{7}{2}x^{5/2} \end{aligned}$$

7. Use the Product Rule with $f(x) = \sin(x)$ and $g(x) = \sqrt{x}$.

$$f(x) = \sin(x) \qquad f'(x) = \cos(x)$$
$$g(x) = \sqrt{x} \qquad g'(x) = \frac{1}{2\sqrt{x}}$$

Then

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= \sin(x) \cdot \frac{1}{2\sqrt{x}} + \sqrt{x} \cdot \cos(x) \\ &= \frac{\sin(x)}{2\sqrt{x}} + \sqrt{x}\cos(x) \end{aligned}$$

8. Use the Quotient Rule.

$$f(x) = ax + b \qquad f'(x) = a$$
$$g(x) = ax - b \qquad g'(x) = a$$

$$\begin{aligned}
\frac{d}{dx}\frac{ax+b}{ax-b} &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \\
&= \frac{(ax-b)\cdot a - (ax+b)\cdot a}{(ax-b)^2} \\
&= \frac{a^2x - ab - (a^2x + ab)}{(ax-b)^2} \\
&= \frac{a^2x - ab - a^2x - ab}{(ax-b)^2} \\
&= -\frac{2ab}{(ax-b)^2}
\end{aligned}$$

9. Use the Product Rule with $f(x) = \sin(x)$ and $g(x) = \cos x$.

$$f(x) = \sin(x) \qquad f'(x) = \cos(x)$$
$$g(x) = \cos(x) \qquad g'(x) = -\sin(x)$$

Then

$$\begin{aligned}
h'(x) &= f(x)g'(x) + g(x)f'(x) \\
&= \sin(x)(-\sin(x)) + \cos(x)\cdot\cos(x) \\
&= \cos^2(x) - \sin^2(x)
\end{aligned}$$

10. Use the Chain Rule.

$$f(x) = \sqrt{x} \qquad f'(x) = \frac{1}{2\sqrt{x}}$$
$$g(x) = 3x - 5 \qquad g'(x) = 3$$

Then

$$\begin{aligned}
h'(x) &= f'(g(x))g'(x) \\
&= \frac{1}{2\sqrt{g(x)}}\cdot 3 \\
&= \frac{3}{2\sqrt{3x-5}}.
\end{aligned}$$

11. Use the Chain Rule.

$$f(x) = \cos(x) \qquad f'(x) = -\sin(x)$$
$$g(x) = x^3 \qquad g'(x) = 3x^2$$

Then

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= -\sin(g(x)) \cdot 3x^2 \\ &= -3x^2 \sin(x^3). \end{aligned}$$

12. Use the Chain Rule. Remember the notation: $\cos^3(x) = (\cos(x))^3$.

$$\begin{aligned} f(x) &= x^3 & f'(x) &= 3x^2 \\ g(x) &= \cos(x) & g'(x) &= -\sin(x) \end{aligned}$$

Then

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= 3(g(x))^2(-\sin(x)) \\ &= -3\sin(x)\cos^2(x). \end{aligned}$$

13. Write $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and use the Quotient Rule.

$$\begin{aligned} f(x) &= \sin(x) & f'(x) &= \cos(x) \\ g(x) &= \cos(x) & g'(x) &= -\sin(x) \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}\frac{\sin(x)}{\cos(x)} &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{\cos(x) \cdot \cos(x) - \sin(x)(-\sin(x))}{(\cos(x))^2} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} \\ &= \sec^2(x). \end{aligned}$$

Here, we used the identity $\cos^2(x) + \sin^2(x) = 1$ and the definition $\sec(x) = \dfrac{1}{\cos(x)}$.

14. Rewrite $\dfrac{\sin(x)}{x^{-3}} = x^3\sin(x)$ and use the Product Rule.

$$\begin{aligned} f(x) &= x^3 & f'(x) &= 3x^2 \\ g(x) &= \sin(x) & g'(x) &= \cos(x) \end{aligned}$$

Then

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= x^3 \cdot \cos(x) + \sin(x) \cdot 3x^2 \\ &= x^3 \cos(x) + 3x^2 \sin(x) \\ &= x^2(x \cos(x) + 3 \sin(x)). \end{aligned}$$

You do not have to factor out in the last step, but be aware that when using other resources, answers may be written this way.

15. (a) Use the Chain Rule.
    (b) Rewrite as $3x^{-7}$ and use the Power Rule.
    (c) Rewrite as $x^2 \cos(x)$ and use the Product Rule.
    (d) Combine exponents to get $x^{5/2}$ and use the Product Rule.
    (e) Simplify by dividing and use the Power Rule.
    (f) Use the Quotient Rule.
    (g) Use the Chain Rule.
    (h) Rewrite as $3x^4$ and use the Power Rule.

Chapter 4

The Second Derivative

4.1 What happens when $f'(x) = 0$?

In Chapter 2, we began our study of the algebra and geometry of derivatives. We saw that:

> If $f'(x) > 0$ for some value of x, then the function $f(x)$ is increasing at that value of x.
>
> If $f'(x) < 0$ for some value of x, then the function $f(x)$ is decreasing at that value of x.

But what happens when $f'(x) = 0$? This is a more complicated scenario, illustrated in Figure 4.1.

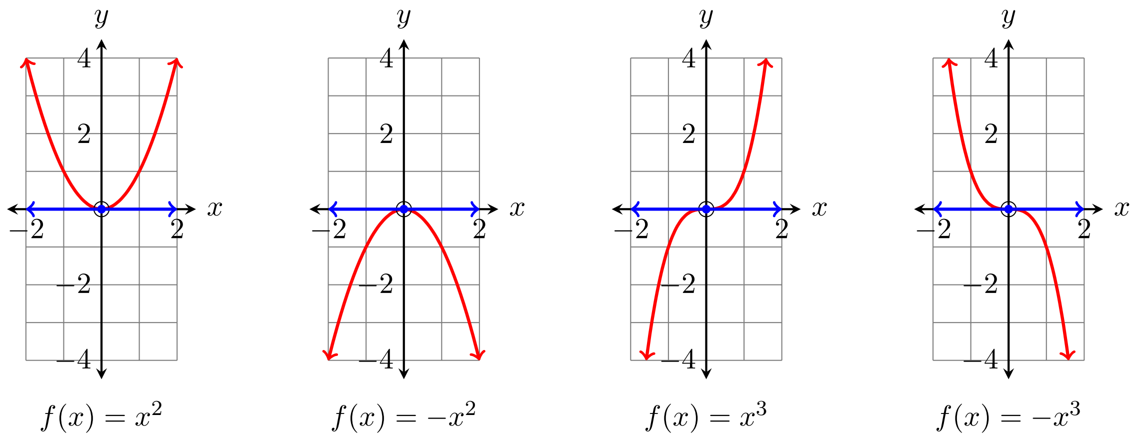


Figure 4.1: What can happen when $f'(x) = 0$.

Let's look at these graphs in detail. In each graph, the blue line is tangent at $x = 0$, and is a horizontal line with slope 0, and so $f'(0) = 0$ in each case.

1. For $f(x) = x^2$, we have what is called a **local minimum** at $x = 0$. This means as we go from left to right, the function decreases until it hits $(0, 0)$, and then starts increasing.
2. For $f(x) = -x^2$, we have what is called a **local maximum** at $x = 0$. This means as we go from left to right, the function increases until it hits $(0, 0)$, and then starts decreasing.
3. For $f(x) = x^3$, we have what is called an **inflection point,** or a **point of inflection.** In this case the function keeps increasing as we pass through $(0, 0)$.
4. For $f(x) = -x^3$, we also have an inflection point, but the function keeps decreasing as we pass through $(0, 0)$.

How do we know which is which? Yes, we can look at the graph. But to decide *without* a graph, we have to use calculus. In this section, we'll learn to use **sign charts** for $f'(x)$ to make this decision, and in the next section, we'll look at how to use **second derivatives.**

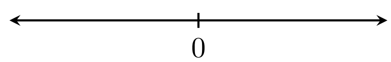
Making Sign Charts

We'll first look at the far left graph in Figure 4.1, $f(x) = x^2$. To make a sign chart for $f'(x)$:

1. Find all values of x where $f'(x) = 0$;
2. Plot these values on a number line;
3. This divides the line into intervals – choose *one* point from each interval (one that is easy to evaluate) and evaluate $f'(x)$; if $f'(x) > 0$, write "+" over the interval, and if $f'(x) < 0$, write "−" above the interval. We'll first go through these steps, and then interpret the results.

Since $f(x) = x^2$, then using the Power Rule, we get $f'(x) = 2x$.

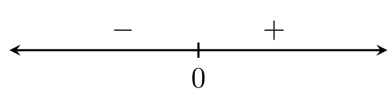
1. If $f'(x) = 2x = 0$, then $x = 0$.
2. This gives the following number line:



3. Now choose one value from each interval. Easy values are $x = -1$ and $x = 1$.

$$\begin{aligned} f'(-1) &= 2(-1) \\ &= -2 \\ &< 0 \\ f'(1) &= 2(1) \\ &= 2 \\ &> 0. \end{aligned}$$

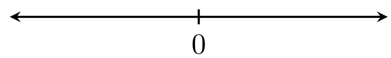
This yields the following number line:



How do we interpret this? On the interval $(-\infty, 0)$, $f'(x)$ is negative, and so we know the function is decreasing on this interval. But on $(0, \infty)$, we see that $f'(x)$ is positive, and so the function is increasing on this interval. Because we go from decreasing to increasing as we pass through $x = 0$, this means there must be a local minimum at $x = 0$. This can be confirmed by looking at the graph.

Now let's look at the rightmost graph in Figure 4.1, $f(x) = -x^3$. We'll make a sign chart here as well. Since $f(x) = -x^3$, then using the Power Rule, we get $f'(x) = -3x^2$.

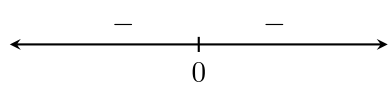
1. If $f'(x) = -3x^2 = 0$, then $x = 0$.
2. This gives the following number line:



3. Now choose one value from each interval. Easy values are $x = -1$ and $x = 1$.

$$
\begin{aligned}
f'(-1) &= -3(-1)^2 \\
&= -3 \\
&< 0 \\
f'(1) &= -3(1)^2 \\
&= -3 \\
&< 0.
\end{aligned}
$$

This yields the following number line:



How do we interpret this? On the interval $(-\infty, 0)$, $f'(x)$ is negative, and so we know the function is decreasing on this interval. But on $(0, \infty)$, we see that $f'(x)$ is also negative, and so the function is decreasing on this interval as well. Because we continuously decrease as we pass through $x = 0$, this means there must be an inflection point at $x = 0$. This can also be confirmed by looking at the graph.

The algebra for these examples was fairly easy, but the point is to introduce the concepts. This is one more example of a recurring theme: we make informal observations about a function by looking at its graph, and then we back up our observations using calculus.

Example 1

Now we'll look at a little more complicated example. Consider the function $f(x) = x^3 - 3x + 2$, shown in Figure 4.2.

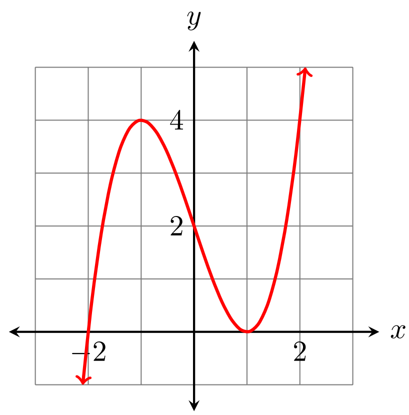


Figure 4.2: Graph of $f(x) = x^3 - 3x + 2$.

It looks like there is a local maximum at $x = -1$ and a local minimum at $x = 1$. Let's verify this by making a sign chart.

1. Since $f(x) = x^3 - 3x + 2$, then using the Power Rule,

$$f'(x) = 3x^2 - 3.$$

Then solving $f'(x) = 0$:

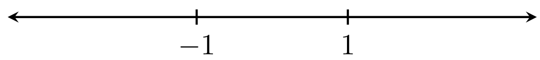
$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - 3 &= 0 \\ 3(x^2 - 1) &= 0 \\ 3(x+1)(x-1) &= 0 \\ x &= 1 \\ x &= -1 \end{aligned}$$

This isn't the only way to solve. You can do the following.

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - 3 &= 0 \\ 3x^2 &= 3 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

WARNING!!! Be very careful if you do this. A common mistake is to forget $x = -1$ when doing it this way. That is, you just go from $x^2 = 1$ to $x = 1$ and leave out the other value of x. I've seen this happen **many** times.

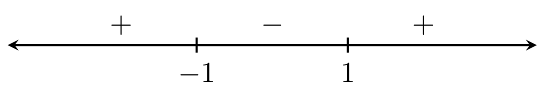
2. This gives the following number line:



3. Now choose one value from each interval. Easy values are $x = -2$, $x = 0$, and $x = 2$.

$$\begin{aligned} f'(-2) &= 3((-2)^2) - 3 \\ &= 9 \\ &> 0 \\ f'(0) &= 3(0^2) - 3 \\ &= -3 \\ &< 0 \\ f'(2) &= 3(2^2) - 3 \\ &= 9 \\ &> 0. \end{aligned}$$

This yields the following number line:



So at $x = -1$, the graph changes from increasing to decreasing, and so there is a local maximum there. And at $x = 1$, the graph changes from decreasing to increasing, and so there is a local minimum there.

ASSESSMENT EXPECTATION: When asked to find local maxima and minima (plurals of maximum and minimum), ALWAYS include the y-values and write your answer as a point. Since $f(-1) = 4$ and $f(1) = 0$, you would say there is local maximum at $(-1, 4)$ and a local minimum at $(1, 0)$.

Summary:

> If $f'(x) > 0$ for some value of x, then the function $f(x)$ is increasing at that value of x.
>
> If $f'(x) < 0$ for some value of x, then the function $f(x)$ is decreasing at that value of x.
>
> If $f'(x) = 0$, there may be a local minimum, a local maximum, or an inflection point (determined by making a sign chart).

Homework

1. Consider the graph of $f(x) = x^3 - 12x - 3$. By making a sign chart for $f'(x)$, find all local minima and maxima. Visually verify this by graphing on desmos.

2. Consider the graph of $f(x) = \sin(x)$ on the interval $[0, 2\pi]$. By making a sign chart for $f'(x)$, find all local minima and maxima. Visually verify this by graphing on desmos.

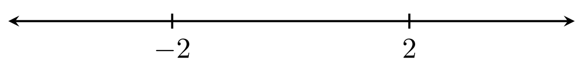
Solutions

1. Since $f(x) = x^3 - 12x - 3$, then $f'(x) = 3x^2 - 12$.

   (a) To make a sign chart, we need to know where $f'(x) = 3x^2 - 12 = 0$.

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - 12 &= 0 \\ 3(x^2 - 4) &= 0 \\ 3(x+2)(x-2) &= 0 \\ x &= 2 \\ x &= -2 \end{aligned}$$

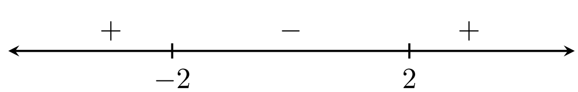
   (b) This gives the following number line:



   (c) Now choose one value from each interval. Easy values are $x = -3$, $x = 0$, and $x = 3$.

$$\begin{aligned} f'(-3) &= 3((-3)^2) - 12 \\ &= 15 \\ &> 0 \\ f'(0) &= 3(0^2) - 12 \\ &= -12 \\ &< 0 \\ f'(2) &= 3(3^2) - 12 \\ &= 15 \\ &> 0. \end{aligned}$$

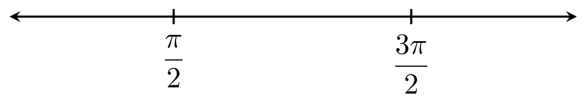
   This yields the following number line:



Because $f'(x)$ goes from $+$ to $-$ at -2, $f(x)$ increases and then decreases. Since $f(-2) = 13$, there is a local maximum at $(-2, 13)$. Because $f'(x)$ goes from $-$ to $+$ at 2, $f(x)$ decreases and then increases. Since $f(2) = -19$, there is a local minimum at $(2, -19)$.

2. To make a sign chart, we need to know where $\cos(x) = 0$ on the interval $[0, 2\pi]$. We can look at the unit circle and see where the x-value is 0. This occurs when $x = \pi/2$ and $x = 3\pi/2$.

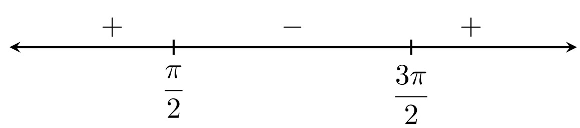
   (a) This gives the following number line:

   (b) Looking at easy values in each interval, we choose $x = \frac{\pi}{4}, \pi, \frac{7\pi}{4}$. Then

$$\begin{aligned} f'(\pi/4) &= \cos(\pi/4) \\ &= 1/\sqrt{2} \\ &> 0 \\ f'(\pi) &= \cos(\pi) \\ &= -1 \\ &< 0 \\ f'(7\pi/4) &= \cos(7\pi/4) \\ &= 1/\sqrt{2} \\ &> 0 \end{aligned}$$

   This yields the following number line:

   Since $f'(x)$ goes from $+$ to $-$ at $\pi/2$, $f(x)$ increases and then decreases. So there is a local maximum at $(\pi/2, 1)$. Since $f'(x)$ goes from $-$ to $+$ at $3\pi/2$, $f(x)$ decreases and then increases. So there is a local minimum at $(3\pi/2, -1)$.

4.2 The Geometry of Second Derivatives

In our discussion of the geometry of the first derivative, we saw that given a function $f(x)$, the graph of the function is increasing at points where $f'(x) > 0$, and decreasing at points where $f'(x) < 0$. But when $f'(x) = 0$, there were three possibilities: a local minimum, a local maximum, or an inflection point. Now if you have a graph, you can just look at it and see which case applies. Here, we will learn another way how to figure this out using calculus.

We will need the **second derivative** here, which is just the derivative *of the derivative.* Let's look at a few examples.

Example 1

Suppose $f(x) = x^4 - 3x^2$. Using the power rule, we get

$$f'(x) = 4x^3 - 6x.$$

What if we take the derivative again? We get

$$\begin{aligned} \frac{d}{dx} f'(x) &= \frac{d}{dx}(4x^3 - 6x) \\ &= 12x^2 - 6, \end{aligned}$$

and write

$$f''(x) = 12x^2 - 6.$$

Sometimes you will see the notation

$$\frac{d^2}{dx^2} f(x) = 12x^2 - 6,$$

although we won't be using this notation – $f''(x)$ is much easier to use.

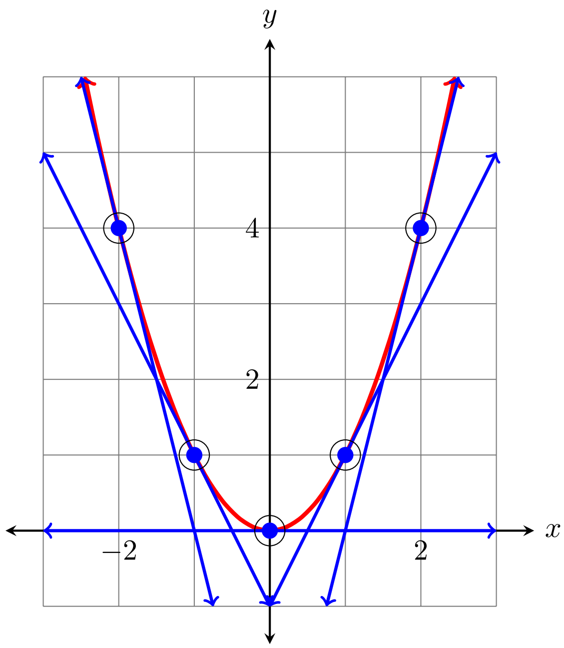
Example 2

Suppose $f(x) = \sin(x)$. Then $f'(x) = \cos(x)$, and

$$\begin{aligned} f''(x) &= \frac{d}{dx}\cos(x) \\ &= -\sin(x). \end{aligned}$$

Example 3

So finding the second derivative is just a matter of taking the derivative twice in a row. But why would we want to do this? Let's look at an example.



x	$f'(x)$	Eqn. of tangent
-2	-4	$y = -4x - 4$
-1	-2	$y = -2x - 1$
0	0	$y = 0$
1	2	$y = 2x - 1$
2	4	$y = 4x - 4$

Figure 4.3: Graph of $f(x) = x^2$ (left) with table of values of $f'(x)$ (right). For an interactive version of this graph, visit desmos.com.

You can see several tangent lines graphed in Figure 4.3. In the table on the right, you can see several values of $f'(x)$. What do we notice? That the slopes of the tangent lines – the values of $f'(x)$ – are *increasing* as we go from left to right.

So $f'(x)$ is a function which is increasing. Remember that when a function is increasing, its derivative is positive. Since $f''(x)$ is the derivative of $f'(x)$, this means that

$$\frac{d}{dx} f'(x) > 0,$$
$$f''(x) > 0.$$

Geometrically, we say that when $f''(x) > 0$, the function is **concave up,** meaning essentially, that the function "opens upward." This can happen in three ways, shown in Figure 4.4.

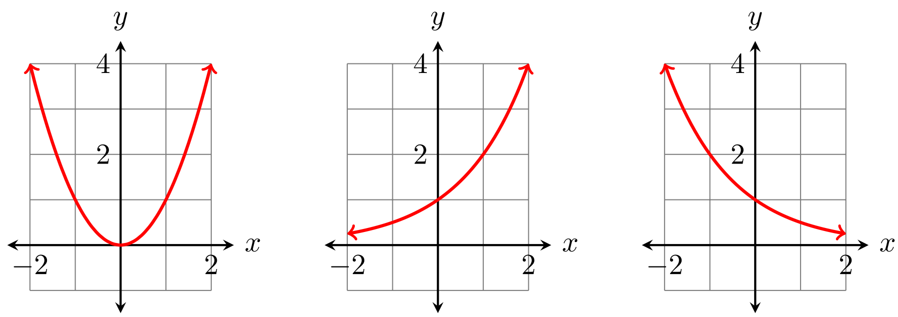


Figure 4.4: Graph of $f(x) = x^2$ (left), $f(x) = 2^x$ (middle), and $f(x) = 2^{-x}$ (right).

There may be a local minimum involved, as with $f(x) = x^2$ (left in Figure 4.4). But there may be no minimum at all. The function $f(x) = 2^x$ is always increasing and concave up (middle graph), but $f(x) = 2^{-x}$ is always decreasing and concave up (right graph). So you can see why the first derivative cannot tell us about concavity: a concave up graph could be increasing, decreasing, or both. That's why we need the second derivative.

Wherever $f''(x) > 0$, the graph of the function is concave up.

When $f''(x) < 0$, we say that the function is **concave down.** We won't repeat the previous analysis since it is very similar. Instead, we'll jump to some examples.

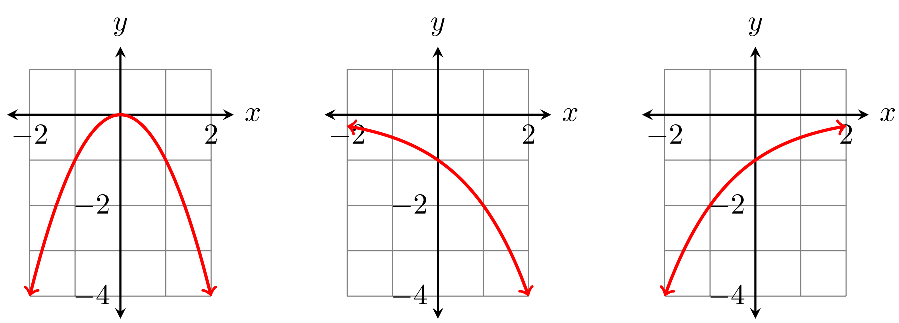


Figure 4.5: Graph of $f(x) = -x^2$ (left), $f(x) = -2^x$ (middle), and $f(x) = -2^{-x}$ (right).

Wherever $f''(x) < 0$, the graph of the function is concave down.

Example 4

The graphs of most functions, though, are partly concave up, and partly concave down. We'll look at such an example, $f(x) = \frac{1}{x}$, and apply what we just learned.

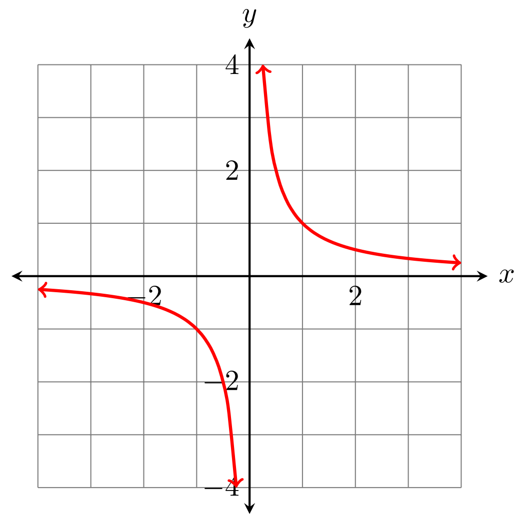


Figure 4.6: Graph of $f(x) = \frac{1}{x}$

Note that $f(x)$ is not defined when $x = 0$. Now let's find $f''(x)$. Remember, we don't need the quotient rule here because we can rewrite the function as $f(x) = x^{-1}$.

$$\begin{aligned} f(x) &= x^{-1} \\ f'(x) &= -x^{-1-1} \\ &= -x^{-2} \\ f''(x) &= -(-2)x^{-2-1} \\ &= 2x^{-3} \end{aligned}$$

So $f''(x) = \frac{2}{x^3}$. When $x > 0$, $f''(x) > 0$ since $(+)(+)(+) = (+)$ on the denominator. Hence the graph is concave up on the interval $(0, \infty)$.

But when $x < 0$, then $f''(x) < 0$ since $(-)(-)(-) = (-)$ on the denominator. Thus the graph is concave down on the interval $(-\infty, 0)$.

Remember that, in general, we use the first derivative, $f'(x)$, to determine where the function is increasing or decreasing. We now know that we use the second derivative, $f''(x)$, to determine where the function is concave up and where it's concave down.

Sometimes it gets confusing to remember what to use – $f(x)$, $f'(x)$, or $f''(x)$ – in a given problem. So here's a summary – so important, it needs to be double-boxed!

What...	...it's used for
$f(x)$	Finding y-values
$f'(x)$	Increasing/decreasing; minimum/maximum
$f''(x)$	Concave up/down; inflection points

The only thing left to discuss is the case when $f''(x) = 0$. Like the case when $f'(x) = 0$, there are four possibilities. These are shown in Figure 4.7.

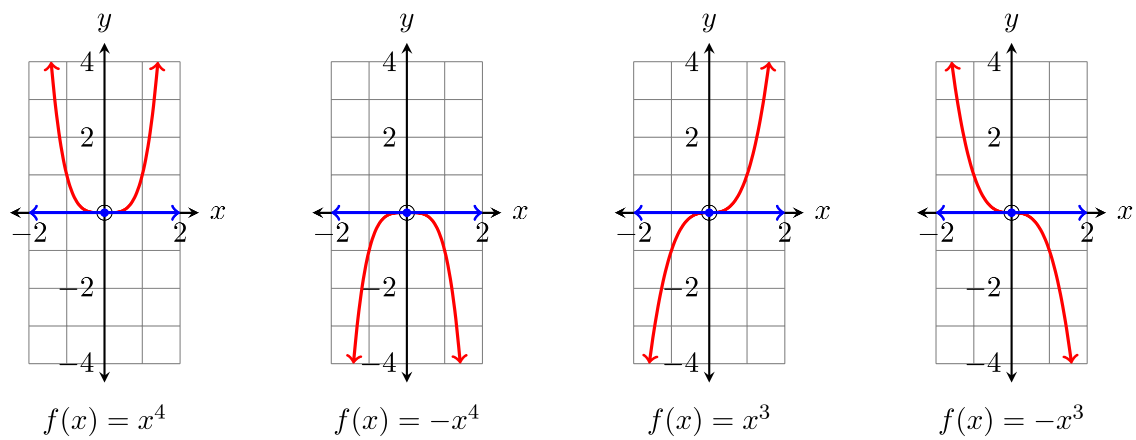


Figure 4.7: Possible behavior when $f''(x) = 0$.

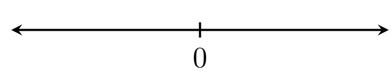
But if we don't have a graph, how can we figure out which is which? We can use a sign chart for $f''(x)$.

We'll first look at the far left graph in Figure 4.7, $f(x) = x^4$. To make a sign chart for $f''(x)$, we first calculate $f''(x)$ when $f(x) = x^4$.

$$\begin{aligned} f'(x) &= 4x^3 \\ f''(x) &= 4 \cdot 3x^2 \\ &= 12x^2. \end{aligned}$$

1. If $f''(x) = 12x^2 = 0$, then $x = 0$.

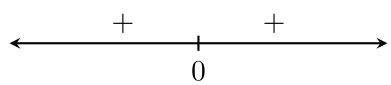
2. This gives the following number line:



3. Now choose one value from each interval. Easy values are $x = -1$ and $x = 1$.

$$\begin{aligned} f''(-1) &= 12(-1)^2 \\ &= 12 \\ &> 0 \\ f''(1) &= 12(1^2) \\ &= 12 \\ &> 0. \end{aligned}$$

This yields the following number line:



How do we interpret this number line? Here is an important definition.

An **inflection point** is a point where a graph *changes* concavity – in other words, it goes from being concave up to concave down, or from concave down to concave up.

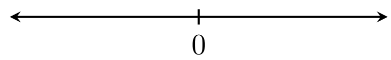
In our example, the graph is concave up on $(-\infty, 0)$ (since $f''(x) > 0$ there), and is *also* concave up on $(0, \infty)$. Thus, the graph does *not* change concavity. Since it is concave up on *both* sides of 0, there is a local minimum at $x = 0$. We can see this by looking at the graph, of course, but a sign chart is necessary when you don't have a graph.

Let's look at another example, this time $f(x) = -x^3$.

1. Since $f(x) = -x^3$, then

$$\begin{aligned} f'(x) &= -3x^2 \\ f''(x) &= -3 \cdot 2x \\ &= -6x. \end{aligned}$$

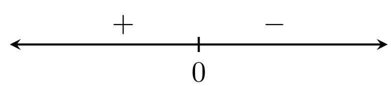
2. This gives the following number line:



3. Now choose one value from each interval. Easy values are $x = -1$ and $x = 1$.

$$\begin{aligned} f''(-1) &= -6(-1) \\ &= 6 \\ &> 0 \\ f''(1) &= -6(1) \\ &= -6 \\ &< 0. \end{aligned}$$

This yields the following number line:



Notice that $f(x)$ is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. Since the graph *changes* concavity at $x = 0$, then this is an inflection point. Again, this is clear from the graph – but without a graph, you can still determine whether $x = 0$ is an inflection point or not using a sign graph.

Example 5

The sign graphs above were fairly simple. Let's look at a more involved example, using $f(x) = x^4 - 6x^2$.

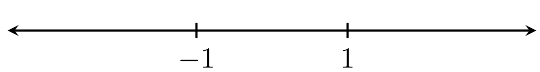
1. Since $f(x) = x^4 - 6x^2$, then

$$\begin{aligned} f'(x) &= 4x^3 - 12x \\ f''(x) &= 12x^2 - 12. \end{aligned}$$

Then

$$\begin{aligned} f''(x) &= 0 \\ 12x^2 - 12 &= 0 \\ 12x^2 &= 12 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

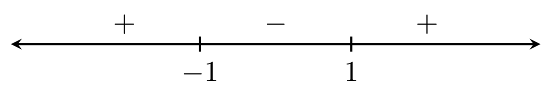
2. This gives the following number line:



3. Now choose one value from each interval. Easy values are $x = -2$, $x = 0$, and $x = 2$.

$$\begin{aligned}
f''(-2) &= 12((-2)^2) - 12 \\
&= 36 \\
&> 0 \\
f''(0) &= 12(0^2) - 12 \\
&= -12 \\
&< 0 \\
f''(2) &= 12(2^2) - 12 \\
&= 36 \\
&> 0.
\end{aligned}$$

This yields the following number line:



So at $x = -1$, the graph changes from concave up to concave down, and at $x = 1$, the graph changes from concave down to concave up. So $x = -1$ and $x = 1$ are both inflection points.

We can see this on the graph below. Keep in mind that we were able to determine this *without* the graph.

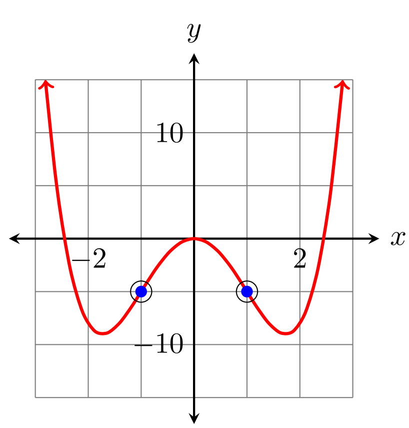


Figure 4.8: Graph of $f(x) = x^4 - 6x^2$.

Homework

1. Let $f(x) = x^7 - \dfrac{1}{x}$. Find $f''(x)$.
2. Let $f(x) = \sqrt{x} + \cos(2x)$. Find $f''(x)$.
3. Let $f(x) = \sin^2(x)$. Find $f''(x)$.
4. Fill in the blank with the best answer.
   (a) When $f''(x) < 0$, the graph of $f(x)$ is ____________________.
   (b) We use a sign graph for $f''(x)$ to determine points of ____________________.
   (c) An inflection point is a point on the graph where the ____________________ changes.
5. Let $f(x) = -x^4$. Create a sign graph for $f''(x)$. Follow the same steps as for the graphs in Figure 4.7.
6. Let $f(x) = -x^4 + 24x^2$. Find all inflection points using a sign chart. (Hint: the "24" should make everything come out nicely.)

Solutions

1.

$$\begin{aligned} f(x) &= x^7 - x^{-1} \\ f'(x) &= 7x^6 + x^{-2} \\ f''(x) &= 42x^5 - 2x^{-3} \end{aligned}$$

2.

$$\begin{aligned} f(x) &= x^{1/2} + \cos(2x) \\ f'(x) &= \frac{1}{2}x^{-1/2} - 2\sin(2x) && \text{Chain rule}: f(x) = \cos(x), g(x) = 2x \\ f''(x) &= -\frac{1}{4}x^{-3/2} - 4\cos(2x) && \text{Chain rule}: f(x) = \sin(x), g(x) = 2x. \end{aligned}$$

3.

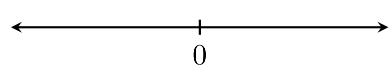
$$\begin{aligned} f(x) &= \sin^2(x) \\ f'(x) &= 2\sin(x)\cos(x) && \text{Chain rule}: f(x) = x^2, g(x) = \sin(x) \\ f''(x) &= (2\sin(x))(-\sin(x)) + (\cos(x))(2\cos(x)) && \text{Product rule}: f(x) = 2\sin(x), g(x) = \cos(x) \\ &= 2(\cos^2(x) - \sin^2(x)) \end{aligned}$$

4. (a) concave down.
   (b) inflection.
   (c) concavity.

5. Let $f(x) = -x^4$. To make a sign chart for $f''(x)$:

$$\begin{aligned} f'(x) &= -4x^3 \\ f''(x) &= -4 \cdot 3x^2 \\ &= -12x^2. \end{aligned}$$

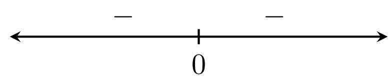
   (a) If $f''(x) = -12x^2 = 0$, then $x = 0$.
   (b) This gives the following number line:



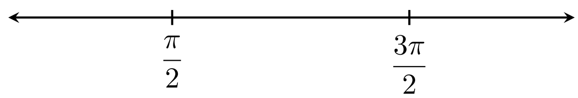
   (c) Now choose one value from each interval. Easy values are $x = -1$ and $x = 1$.

$$\begin{aligned} f''(-1) &= -12(-1)^2 \\ &= -12 \\ &< 0 \\ f''(1) &= -12(1^2) \\ &= -12 \\ &< 0. \end{aligned}$$

   This yields the following number line:



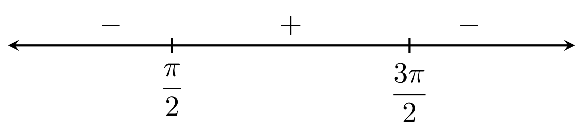
Since the concavity does *not* change, there is no inflection point. Since the graph stays concave down, there is a local maximum at $x = 0$.



Looking at easy values in each interval, we choose $x = \frac{\pi}{4}, \pi, \frac{7\pi}{4}$. Then

$$\begin{aligned} f''(\pi/4) &= -\cos(\pi/4) \\ &= -1/\sqrt{2} \\ &< 0 \\ f''(\pi) &= -\cos(\pi) \\ &= 1 \\ &> 0 \\ f''(7\pi/4) &= -\cos(7\pi/4) \\ &= -1/\sqrt{2} \\ &< 0 \end{aligned}$$

This yields the following number line:



Since concavity changes at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, these must be inflection points.

6. Let $f(x) = -x^4 + 24x^2$.

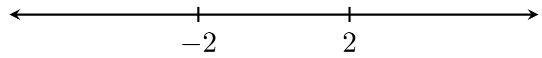
   (a) Since $f(x) = -x^4 + 24x^2$, then

$$\begin{aligned} f'(x) &= -4x^3 + 48x \\ f''(x) &= -12x^2 + 48. \end{aligned}$$

   Then

$$\begin{aligned} f''(x) &= 0 \\ -12x^2 + 48 &= 0 \\ -12x^2 &= -48 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

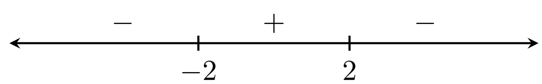
(b) This gives the following number line:



(c) Now choose one value from each interval. Easy values are $x = -3$, $x = 0$, and $x = 3$.

$$\begin{aligned} f''(-3) &= -12((-3)^2) + 48 \\ &= -60 \\ &< 0 \\ f''(0) &= -12(0^2) + 48 \\ &= 48 \\ &> 0 \\ f''(3) &= -12(3^2) + 48 \\ &= -60 \\ &< 0. \end{aligned}$$

This yields the following number line:



So at $x = -2$, the graph changes from concave down to concave up, and at $x = 2$, the graph changes from concave up to concave down. So $x = -2$ and $x = 2$ are both inflection points.

Chapter 5

Exponentials and Logarithms

5.1 Exponential Functions and e

To begin, review Exponential Functions in the pdf linked to from the website, on pp. 417–419.

There is a particular choice of base b which we use a *lot* in calculus – the base e. So let's look at why this particular number is important. You want to think of e like π – it is an irrational number which cannot be represented by a fraction. The more you work problems with e, the more you'll get used to it.

You'll need to go to this desmos page on Exponential Derivatives, since the notes here will be referring to the graphs there. You'll see a graph (in red) of $y = b^x$, with a slider you can move to change the base b. Then you also see a graph (in blue) of $f'(x)$.

Right now, we don't have a formula for $f'(x)$. Remember, to use the Power Rule, the x has to be in the *base,* but with $y = b^x$, the x is in the *exponent.* Moreover, b^x cannot be written as product or quotient, so we can't apply these rules, and we can't use the Chain Rule either.

But you should notice one thing. The derivative of an exponential function looks like *another* exponential function. Why should this be?

To help you see this, please go to the desmos page Exponential Tangents. Here, you see the graph of $y = 2^x$. As you move the slider for a, you'll see the tangent line at a along with its slope. (Don't worry about the complicated looking second formula; we won't be needing this.)

So let's take a look at these graphs. As we look at the graph of $y = 2^x$ as we go from -5 to 5, you notice that it starts off very small, gradually increases, and then begins to increasing more rapidly the further right you get. But this is the *same* behavior we notice with the slopes of the tangent lines. The slopes (in blue) start off small, gradually increase, but increase even faster the further we go to the right.

Now a graphical observation is not a mathematical *proof* that the derivative of an exponential function is another exponential, but it does turn out to be true. These desmos graphs are just meant to show why it makes graphical sense. We will not need a formal proof.

So if the derivative of $f(x) = 2^x$ is another exponential function, just *what* exponential function is it? We won't completely answer this question right now, but at least we'll get a good start on it.

Now navigate back to Exponential Derivatives. Remember, the red graph is $y = b^x$, and the derivative is the blue graph. Now starting at the left of the slider for b, you will notice that the derivative graph is *below* the exponential graph. As you move the slider to the right, you should see that the derivative graph moves up closer to the exponential graph. As b approaches about 2.7, you should notice that the derivative graph crosses over and is now *above* the exponential graph.

What is happening around $b = 2.7$? There is *exactly* one value of b where the derivative graph is *exactly* on top of the exponential graph. For lesser values, the derivative graph is below the exponential graph, and for greater values, the derivative graph is above the exponential graph. The value of b where the two graphs are the same is called e, where $e \approx 2.71828$. The number e is an irrational number; there is no simple formula for it, just like π.

In terms of calculus, we can summarize these observations as follows. Note the important double box!

$$\boxed{\frac{d}{dx}e^x = e^x}$$

This means that the derivative graph of e^x is *exactly* the same graph as e^x. Don't forget this formula! Remember, we need this formula because we can't use the Power Rule – the x is not in the base, it's in the exponent.

Example 1

Let $h(x) = 4e^{0.5x}$. Find $h'(x)$.

Our formula applies *only* if the exponent of e is just x. Since the exponent is different here, we need the Chain Rule. We will use

$$f(x) = 4e^x, \quad f'(x) = 4e^x,$$
$$g(x) = 0.5x, \quad g'(x) = 0.5$$

$$\begin{aligned} h(x) &= 4e^{0.5x} \\ h'(x) &= 4f'(g(x))g'(x) \\ &= 4e^{g(x)}(0.5) \\ &= 2e^{0.5x} \end{aligned}$$

Example 2

You make a purchase of \$1000 on your credit, which has an APR of 20%. After one month, how much interest will you have to pay on the \$1000 if you do not pay off your card in full? Assume a month has 30 days.

Let's review the compound interest formula for daily compounded interest.

$$\text{Balance} = \text{Principal}\left(1 + \frac{\text{Interest Rate}}{365}\right)^{\text{Days}}$$

Plugging in, we get

$$\text{Balance} = \$1000\left(1 + \frac{0.20}{365}\right)^{30} \approx \$1016.57.$$

So you are paying \$16.57 in interest.

Exponential functions can be used to approximate interest compounded over short times, like days. The formula is

$$\text{Balance} = \text{Principal} \times e^{\text{Interest Rate} \times \text{Years}}$$

Here, we use 1/12 for Years since interest is accruing for just one month.

$$\text{Balance} = \$1000 \times e^{0.20 \times 1/12} \approx \$1016.81.$$

With this formula, you are paying \$16.81 in interest, which is off by only a few cents.

Example 3

One application of exponential functions is *bacterial growth.* This is a good model for the initial growth spurt when a culture of bacteria is started in a Petri dish. After a while, though, the Petri dish begins to fill up and the growth rate slows down. To model the slowing part down as well, you'll need to wait until Calculus II.

You'll need to get used to using the variable t for exponential growth, since t is the variable most used for time. So suppose a population of bacteria is modeled by

$$P(t) = 5000e^{0.01t},$$

where P is the population at time t, which is given in hours. Let's look at a few questions.

1. What is the initial population?
2. What is the population after 10 hours?
3. At what rate is the population increasing at 10 hours?

Solutions:

1. The term *initial population* always refers to the time $t = 0$. $P(0) = 5000e^{0.01(0)} = 5000$, so the initial population is 5000 bacteria.
2. After 10 hours, the population is $P(10) = 5000e^{0.01(10)} \approx 5525.85$. (You should have a key on your calculator which calculates e^x.) Since you can't have a fractional number of bacteria, we usually round up and say the population is 5526.
3. Since we're asking for a rate, we need the derivative – just like the velocity is the rate of change of the displacement. We will use the Chain Rule again. We will use the variable t for time.

$$f(t) = 5000e^t, \quad f'(t) = 5000e^t,$$
$$g(t) = 0.01t, \quad g'(t) = 0.01.$$

$$\begin{aligned} P(t) &= 5000e^{0.01t} \\ P'(t) &= 5000f'(g(t))g'(t) \\ &= 5000e^{g(t)}(0.01) \\ &= 50e^{0.01t} \end{aligned}$$

Now use your calculator to see that $P'(10) \approx 55.2585$. So the population increase at 10 hours is approximately 56 bacteria per hour.

Example 4

When calculating derivatives involving e^x, you will almost always need the Chain Rule. Suppose $h(x) = e^{\sin(x)}$. Find $h'(x)$.

To do this, we let

$$f(x) = e^x, \quad f'(x) = e^x,$$
$$g(x) = \sin(x), \quad g'(x) = \cos(x)$$

Then

$$\begin{aligned} h(x) &= e^{\sin(x)} \\ h'(x) &= f'(g(x))g'(x) \\ &= e^{g(x)}(\cos(x)) \\ &= \cos(x)e^{\sin(x)} \end{aligned}$$

Example 5

Suppose $h(x) = e^{x^2}$. Find $h'(x)$. (Note: e^{x^2} means $e^{(x^2)}$, so you cannot simplify using rules of exponents first. This is just something you have to know; just like the square root sign means a $\frac{1}{2}$ power.)

To do this, we let

$$f(x) = e^x, \quad f'(x) = e^x,$$
$$g(x) = x^2, \quad g'(x) = 2x.$$

Then

$$\begin{aligned} h(x) &= e^{x^2} \\ h'(x) &= f'(g(x))g'(x) \\ &= e^{g(x)}(2x) \\ &= 2xe^{x^2} \end{aligned}$$

Homework

1. Suppose $f(x) = 2xe^x$. Find $f'(x)$.
2. Let $f(x) = \dfrac{e^x}{e^x + 1}$.
3. Let $h(x) = e^x \sin(x)$. Find $h'(x)$.
4. Let $h(x) = e^{\sin(x)+\cos(x)}$. Find $h'(x)$.
5. Suppose $g(x) = e^{\sqrt{x}}$.
6. Suppose you make a purchase of \$250 on a credit card with an APR of %18. Using an exponential function, approximate how much interest will have accrued in one month.
7. Suppose a population of bacteria is modeled by

$$P(t) = 4000e^{0.02t},$$

where P is the population at time t, which is given in hours.

(a) What is the initial population?

(b) What is the population after 5 hours?

(c) At what rate is the population increasing at 5 hours?

Solutions:

1. We need the Product Rule here. We'll use

$$f(x) = 2x, \quad f'(x) = 2,$$

$$g(x) = e^x, \quad g'(x) = e^x.$$

Remember the $f(x)$ in the Product Rule is *not* the same as the original $f(x)$. Then

$$\begin{aligned}\frac{d}{dx}2xe^x &= f(x)g'(x) + g(x)f'(x) \\ &= 2xe^x + e^x(2) \\ &= 2xe^x + 2e^x \\ &= 2e^x(x+1).\end{aligned}$$

It is not necessary to factor out the $2e^x$, but that is likely the answer a book or software would give you.

2. We need the Quotient Rule here. We'll use

$$f(x) = e^x, \quad f'(x) = e^x,$$

$$g(x) = e^x + 1, \quad g'(x) = e^x.$$

Also, don't forget that

$$e^x \cdot e^x = e^{x+x} = e^{2x}.$$

Then

$$\begin{aligned}\frac{d}{dx}\frac{e^x}{e^x+1} &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{(e^x+1)e^x - e^x \cdot e^x}{(e^x+1)^2} \\ &= \frac{e^{2x} + e^x - e^{2x}}{(e^x+1)^2} \\ &= \frac{e^x}{(e^x+1)^2}\end{aligned}$$

3. We use the Product Rule here.

$$f(x) = e^x, \quad f'(x) = e^x,$$

$$g(x) = \sin(x), \quad g'(x) = \cos(x).$$

$$\begin{aligned}h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= e^x(\cos(x)) + \sin(x)(e^x) \\ &= e^x(\cos(x) + \sin(x))\end{aligned}$$

4. We use the Chain Rule here.

$$f(x) = e^x, \quad f'(x) = e^x,$$
$$g(x) = \sin(x) + \cos(x), \quad g'(x) = \cos(x) - \sin(x).$$

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= e^{g(x)}(\cos(x) - \sin(x)) \\ &= (\cos(x) - \sin(x))e^{\sin(x)+\cos(x)} \end{aligned}$$

5. We use the Chain Rule here.

$$f(x) = e^x, \quad f'(x) = e^x,$$
$$g(x) = \sqrt{x}, \quad g'(x) = \frac{1}{2\sqrt{x}}.$$

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= e^{g(x)} \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} e^{\sqrt{x}} \end{aligned}$$

6. We use the formula

$$\text{Balance} = \text{Principal} \times e^{\text{Interest Rate} \times \text{Years}}$$

and substitute in the given values.

$$\text{Balance} = \$250 \times e^{0.18 \times 1/12} \approx \$253.78.$$

So you'd pay

$$\$253.78 - \$250 = \$3.78$$

in interest after one month.

7. (a) The term *initial population* refers to the time $t = 0$. $P(0) = 4000e^{0.02(0)} = 4000$, so the initial population is 4000 bacteria.

   (b) After 5 hours, the population is $P(5) = 4000e^{0.02(5)} \approx 4420.68$. Since you can't have a fractional number of bacteria, we usually round up and say the population is 4421.

   (c) Since we're asking for a rate, we need the derivative – just like the velocity is the rate of change of the displacement. We will use the Chain Rule again.

$$f(t) = 4000e^t, \quad f'(t) = 4000e^t,$$
$$g(t) = 0.02t, \quad g'(t) = 0.02.$$

$$\begin{aligned} P(t) &= 4000e^{0.02t} \\ P'(t) &= 4000f'(g(t))g'(t) \\ &= 4000e^{g(t)}(0.02) \\ &= 80e^{0.02t} \end{aligned}$$

   Now use your calculator to see that $P'(5) \approx 88.4137$. So the population increase at 5 hours is approximately 89 bacteria per hour.

5.2 The Natural Logarithm

It turns out that because the exponential function is so important in calculus, so is its inverse, called the **natural logarithm.** Let's first review some concepts from precalculus.

Example 1

A function is said to pass the **horizontal line test** if any horizontal line passes through no more than one point on the graph. For example, $y = x^3$ passes this test because any horizontal line passes through just one point (left of Figure 5.1). But $y = x^2$ fails the horizontal line test because there are horizontal lines passing through two points on the graph (right of Figure 1).

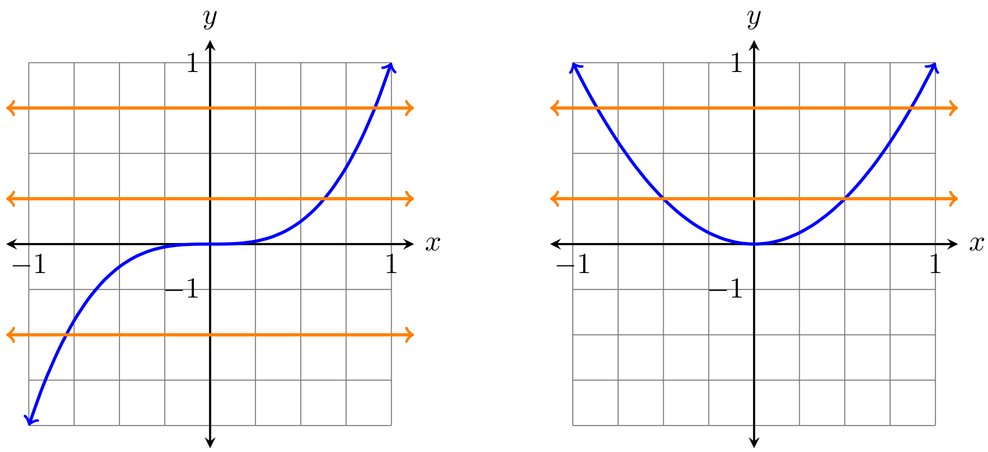


Figure 5.1: Graph of $y = x^3$ passes the horizontal line test (left); graph of $y = x^2$ fails the horizontal line test (right).

When a function passes the horizontal line test, it is said to be **one-to-one,** meaning one y-value can correspond to only one x-value. Such functions are also said to be **invertible.**

Let's start with $y = x^2$ again (red in Figure 5.2). To get the graph of an inverse function, you reflect the graph along the line $y = x$ (green); this is why we switch x and y to solve for the inverse function. The reflected graph is shown in blue. So if a horizontal line goes through two points on a graph (the orange line intersecting the graph of $y = x^2$), when you reflect it a *vertical* line will pass through two points of the inverse graph (the orange line passing through two points on the reflected graph). But a function must pass the vertical line test – one input can not have more than one output. So $y = x^2$ is not invertible because when you reflect the graph, it fails the vertical line test.

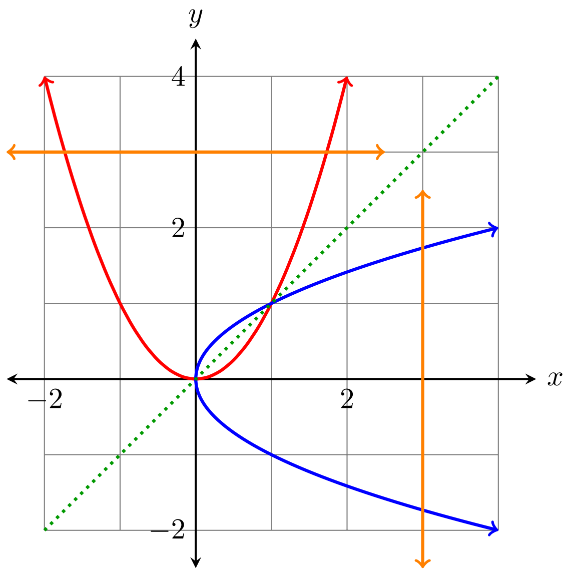


Figure 5.2: Graph of $y = x^2$ (red) and its reflection (blue) along $y = x$.

In order to create a function which *is* invertible, it is sometimes necessary to restrict the domain. As you can see in Figure 5.3, if we restrict the domain to $[0, \infty)$, then the graph of $y = x^2$ *does* pass the horizontal line test, and so we can take its inverse.

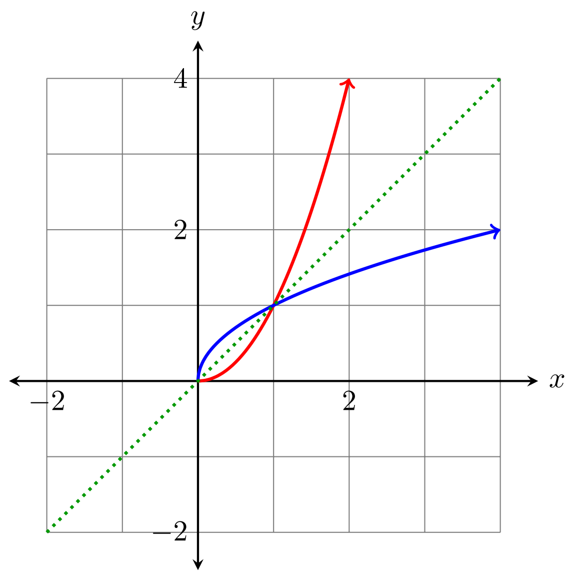


Figure 5.3: Graph of $y = x^2$ with restricted domain (red) and its reflection (blue) along $y = x$.

This is the *geometry* of inverse functions. What about the algebra of inverse functions? If you have an equation of an invertible function, you just switch x and y (which is the algebraic way to reflect along the line $y = x$) and solve for y. Remember that because we restricted the domain, both x and y are positive, so there is no problem taking square roots.

$$\begin{aligned} y &= x^2 && \\ x &= y^2 && \text{switch } x \text{ and } y \\ \sqrt{x} &= y && \text{take square roots} \\ y &= \sqrt{x} && \end{aligned}$$

This means that the function $y = \sqrt{x}$ is the inverse function of $y = x^2$ (with restricted domain).

This example is a review of how to find an inverse function. If you feel like you need to review a bit more, see Section 5.2 on p. 378 of the precalculus text on the website.

The Natural Logarithm

Now let's look at taking the inverse function of $y = e^x$. Note that $y = e^x$ is one-to-one, and so we don't need to worry about restricting the domain.

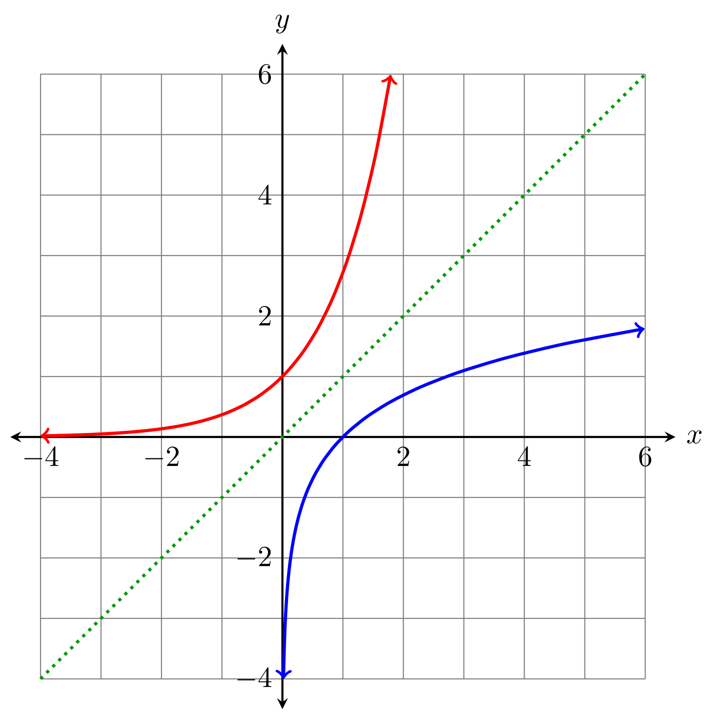


Figure 5.4: Graph of $y = e^x$ (red) and its reflection (blue) along $y = x$.

Graphically, reflecting $y = e^x$ along $y = x$ isn't difficult. It's the algebra which is a bit tricky. Switching x and y gives $x = e^y$, but the problem is that there is no way to solve for y using algebra that we already know.

The first step is giving a name to this inverse function – it's called the *natural logarithm,* and the notation is $y = \ln x$. There is no *formula* for $\ln x$, so any properties of the natural logarithm have to be deduced from properties of exponential functions.

Since $y = e^x$ and $y = \ln x$ are inverse functions, then

$$\boxed{e^{\ln x} = x, \quad \ln(e^x) = x.}$$

This is just like saying that for $y = x^2$ and $y = \sqrt{x}$, whenever $x \geq 0$, we have

$$(\sqrt{x})^2 = x, \quad \sqrt{x^2} = x.$$

What else can we say about the natural logarithm? Let's figure out a useful property of logarithms. If $a, b > 0$, what can we say about $\ln(ab)$? Put $c = \ln(ab)$ and follow along.

$$
\begin{aligned}
\ln(ab) &= c \\
e^{\ln(ab)} &= e^c && \text{substitute into } e^x \\
ab &= e^x && \text{inverse function property} \\
e^{\ln a} \cdot e^{\ln b} &= e^c && \text{inverse function property} \\
e^{\ln a + \ln b} &= e^c && \text{rules of exponents} \\
\ln a + \ln b &= c && e^x \text{ is one-to-one} \\
\ln(ab) &= \ln a + \ln b.
\end{aligned}
$$

The important point here is that we used a rule of exponents to get a rule of logarithms by using the fact that exponential functions and logarithms are inverses of each other.

We won't go through deriving all the properties of natural logarithms, but instead summarize them below.

This property	is valid when...
$e^{\ln a} = a$	$a > 0$
$\ln(e^a) = a$	any a
$\ln(ab) = \ln a + \ln b$	$a, b > 0$
$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$	$a, b > 0$
$\ln(a^m) = m \ln a$	$a > 0$, any m

We remark that $\ln a$ is sometimes called the **logarithm to the base** e **of** a, and is written $\ln a = \log_e a$. We will look at other bases later; for now, our focus is on the natural logarithm.

Before going further, let's do a few examples to review these rules.

1. Find $\ln 1$.

   Method 1: We know that $e^0 = 1$, since any number raised to the 0 power is equal to 1. Since $\ln(e^a) = a$, then $\ln(e^0) = 0$.

   Method 2: We know that $(0, 1)$ is on the graph of every exponential function. Reflecting over the line $y = x$, this means that $(1, 0)$ is on the graph of every logarithmic function, meaning that $\ln 1 = 0$. See Figure 5.4.

2. Find $\ln\left(\frac{1}{e^3}\right)$.

   Method 1: Since $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$, we know (using the previous result) that

   $$\begin{aligned} \ln\left(\frac{1}{e^3}\right) &= \ln 1 - \ln(e^3) \\ &= 0 - \ln(e^3) \\ &= -\ln(e^3). \end{aligned}$$

   But since $\ln(e^a) = a$, we know that $\ln(e^3) = 3$. Thus,

   $$\ln\left(\frac{1}{e^3}\right) = -3.$$

   Method 2: We can write $\frac{1}{e^3} = e^{-3}$ using negative exponents. Thus,

   $$\begin{aligned} \ln\left(\frac{1}{e^3}\right) &= \ln(e^{-3}) \\ &= -3. \end{aligned}$$

Derivative of $\ln x$.

How can we find the derivative of $\ln x$? Using the limit definition is messy – instead we'll use the fact that $\frac{d}{dx}e^x = e^x$ and use the geometry of inverse functions.

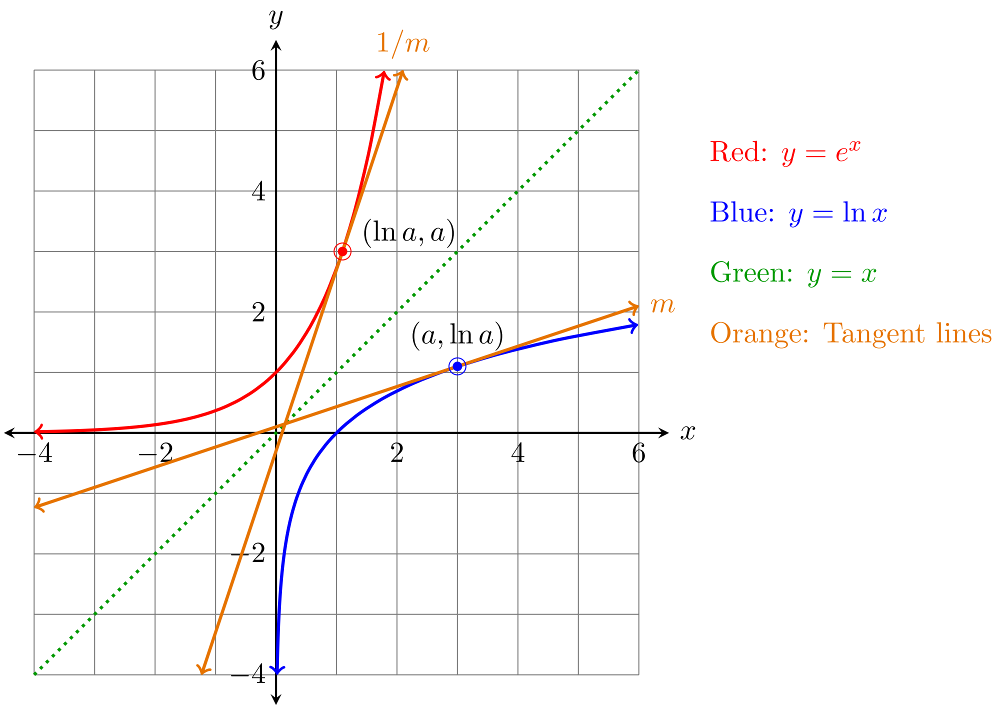


Figure 5.5: Graph of $y = e^x$ (red) and its reflection (blue) along $y = x$.

It looks like there is a *lot* going on in Figure 5.5, so let's look it one piece at a time. We'll start with $x = a$, so that $(a, \ln a)$ is on the graph of $y = \ln x$. You can also see the tangent line at this point.

Now let's reflect across the line $y = x$. Algebraically, this amounts to switching x and y values, so now the point $(\ln a, a)$ is on the graph of $y = e^x$. The tangent line here is also drawn.

What happens when we reflect tangent lines? Suppose that you start with a line with slope m (such as the tangent to $y = \ln x$). Then

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x}.$$

Switching x and y (that is, reflecting the line) gives

$$\frac{\text{change in } x}{\text{change in } y} = \frac{\text{run}}{\text{rise}} = \frac{1}{m}.$$

In other words, when you reflect a line with slope m over $y = x$, the reflected line has the *reciprocal* slope, $\frac{1}{m}$. (Don't confuse this with perpendicular lines, whose slopes are *negative* reciprocals. There is no negative sign here.)

Up to this point, we've just studied the geometry of Figure 5.5. Now it's time to use the fact that $\frac{d}{dx}e^x = e^x$. The point $(\ln a, a)$ is on the graph of $y = e^x$. To find the slope of the tangent line, we plug $x = \ln a$ into the derivative of e^x, which is just e^x. Therefore, using a property of inverse functions, the slope of the tangent line to $y = e^x$ is

$$e^{\ln a} = a.$$

Remember that $e^{\ln a}$ is the slope of the line tangent to $(\ln a, a)$ on the graph of $y = e^x$. Referring back to Figure 5.5, this means that

$$\frac{1}{m} = a,$$

so that

$$m = \frac{1}{a}.$$

So the slope of the tangent line to $y = \ln x$ at $x = a$ is just $\frac{1}{a}$. But the slope of the tangent line is just the derivative, so we have shown that

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

Seems like a lot of work to find a derivative! Finding the inverse function of $y = x^2$ was easy because when we switched x and y to get $x = y^2$, it was easy to solve for y. But for the inverse function of $y = e^x$, there is *no* way to solve $x = e^y$ for y. So we needed to rely heavily on the geometry of inverse functions in order to find the derivative of $y = \ln x$.

Example 2

Suppose $h(x) = \ln(x^3)$. Find $h'(x)$. We can use the Chain Rule here, with

$$f(x) = \ln(x), \quad f'(x) = \frac{1}{x},$$

$$g(x) = x^3, \quad g'(x) = 3x^2.$$

Then

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= \frac{1}{g(x)} \cdot 3x^2 \\ &= \frac{1}{x^3} \cdot 3x^2 \\ &= \frac{3}{x} \end{aligned}$$

It turns out that there is another way to solve this problem. Using a rule of logarithms, we can write

$$h(x) = 3\ln(x).$$

Then we just use the derivative of the logarithm to get $h'(x) = \frac{3}{x}$. This method is simpler, but it does require understanding the rules of logarithms.

Example 3

Suppose $h(x) = \ln(xe^x)$. Find $h'(x)$. Again, let's try the Chain Rule first, with

$$f(x) = \ln(x), \quad f'(x) = \frac{1}{x},$$

$$g(x) = xe^x, \quad g'(x) = xe^x + e^x,$$

where $g'(x)$ was found using the Product Rule.

$$\begin{aligned}
h(x) &= f(g(x)) \\
h'(x) &= f'(g(x))g'(x) \\
&= \frac{1}{g(x)} \cdot (xe^x + e^x) \\
&= \frac{1}{xe^x} \cdot (xe^x + e^x) \\
&= \frac{xe^x}{xe^x} + \frac{e^x}{xe^x} \\
&= 1 + \frac{1}{x}
\end{aligned}$$

Now we'll use the rules of logarithms to find a simpler way. Using two of the rules of logarithms, we can write

$$\begin{aligned}
h(x) &= \ln(xe^x) \\
&= \ln x + \ln(e^x) \\
&= \ln x + x \\
h'(x) &= \frac{1}{x} + 1.
\end{aligned}$$

It is important to note that simplifying using rules of logarithms is not always possible. But when you can apply the rules, very often the process of taking the derivative is much simpler.

Homework

1. Simplify $\ln(e^6)$.
2. If $a = e^b$, then $b =$ ______________ .
3. Explain, in your own words, why, when you reflect a given line across $y = x$, the slope of the reflected line is the reciprocal of the slope of the given line.
4. Find the derivative of $h(x) = \ln\left(\frac{e^x}{x}\right)$ by (1) using the Chain Rule, (2) using rules of logarithms first to simplify.
5. If $h(x) = \ln(\ln(x))$, find $h'(x)$.
6. Find the equation of the tangent line to $y = \ln x$ at $x = 3$. Check that your answer makes sense numerically by looking at Figure 5.5.
7. We see from the graph that $y = \ln x$ is increasing. Show this using calculus. Hint: How do you know that the graph of a function is increasing?
8. We see from the graph that $y = \ln x$ is concave down. Show this using calculus. Hint: How do you know that the graph of a function is concave down?

Solutions

1. 6, since exponential and logarithmic functions are inverses of each other.

2. $\ln a$, since exponential and logarithmic functions are inverses of each other. We often describe this by saying that "a logarithm is an exponent."

3. Answers will be different for everyone.

4. Suppose $h(x) = \ln\left(\frac{e^x}{x}\right)$. First, we'll use the chain rule ($g'(x)$ was found using the Quotient Rule):

$$f(x) = \ln(x), \quad f'(x) = \frac{1}{x}$$

$$g(x) = \frac{e^x}{x}, \quad g'(x) = \frac{xe^x - e^x}{x^2}.$$

Then

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= \frac{1}{g(x)}\left(\frac{xe^x - e^x}{x^2}\right) \\ &= \frac{x}{e^x}\left(\frac{e^x}{x} - \frac{e^x}{x^2}\right) \\ &= \frac{x}{e^x}\cdot\frac{e^x}{x} - \frac{x}{e^x}\cdot\frac{e^x}{x^2} \\ &= 1 - \frac{1}{x}. \end{aligned}$$

Next, we'll use rules of logarithms to simplify first.

$$\begin{aligned} h(x) &= \ln\left(\frac{e^x}{x}\right) \\ &= \ln(e^x) - \ln x \\ &= x - \ln x \\ h'(x) &= 1 - \frac{1}{x} \end{aligned}$$

5. Let $h(x) = \ln(\ln x)$. We use the Chain Rule with

$$f(x) = \ln(x), \quad f'(x) = \frac{1}{x},$$
$$g(x) = \ln(x), \quad g'(x) = \frac{1}{x}.$$

Then

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= \frac{1}{g(x)} \cdot \frac{1}{x} \\ &= \frac{1}{x \ln x}. \end{aligned}$$

6. Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$, so $f'(3) = \frac{1}{3}$, which is the slope m of the tangent line. The point $(x_1, y_1) = (3, \ln 3)$ is also on the tangent line, so we have enough information to find an equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - \ln 3 &= \frac{1}{3}(x - 3) \\ &= \frac{1}{3}x - 1 \\ y &= \frac{1}{3}x - 1 + \ln 3 \\ &\approx \frac{1}{3}x + 0.1 \end{aligned}$$

Looking at Figure 5.5, this make sense. You can see that $\frac{\text{rise}}{\text{run}}$ is about $\frac{1}{3}$, and the y-intercept is just slightly above the origin.

7. Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$. But the domain of $f(x)$ is all numbers $x > 0$. Since $x > 0$, then $\frac{1}{x} > 0$ as well, meaning that the function is always increasing.

8. Continuing from the previous problem,

$$\begin{aligned} f'(x) &= \frac{1}{x} \\ &= x^{-1} \\ f''(x) &= -1 \cdot x^{-2} \\ &= -\frac{1}{x^2}. \end{aligned}$$

Since x^2 is always positive, then $f''(x)$ is always negative. This means that the function is concave down.

Chapter 6

Continuity

6.1 Limits and Continuity

Up to this point, we have used limits to help us define derivatives. The reason we needed the idea of a limit is that we looked at the slopes of secant lines, such as

$$m = \frac{f(x+h) - f(x)}{h}.$$

We wanted to see what happens as $h \to 0$, but we can't just plug in $h = 0$. Or else we'd get $\frac{0}{0}$, which is undefined. So we wrote

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Limits have many other uses in mathematics and calculus. We'll look at three important concepts, illustrated in Figure 6.1. The terms "essential discontinuity," "removable discontinuity," and "continuity" will be explained as we study them in more detail.

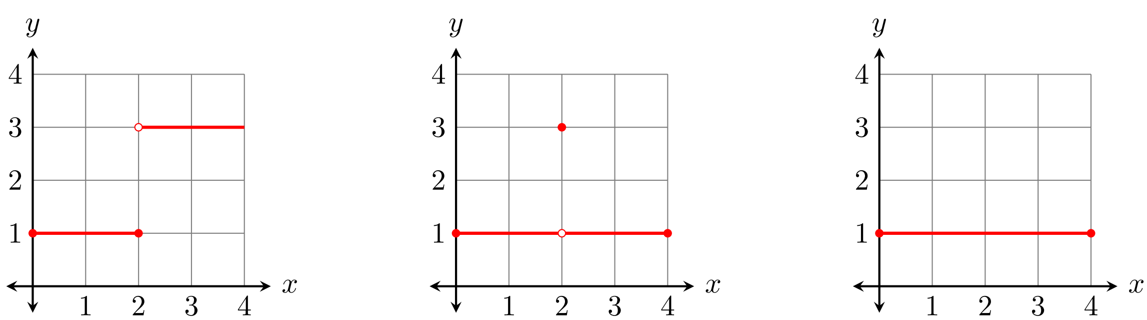


Figure 6.1: Graph of a function $g(x)$ with an essential discontinuity (left), removable discontinuity (middle), and a continuous graph (right).

Example 1

We first look at a function $f(x)$, where x is the number of ounces your letter weighs and $f(x)$ is how much is costs to send your letter. The graph is shown in Figure 6.2. If you mail a first class letter, you pay \$0.60 for the first ounce (up to exactly one ounce), and \$0.24 for each part of an ounce after that.

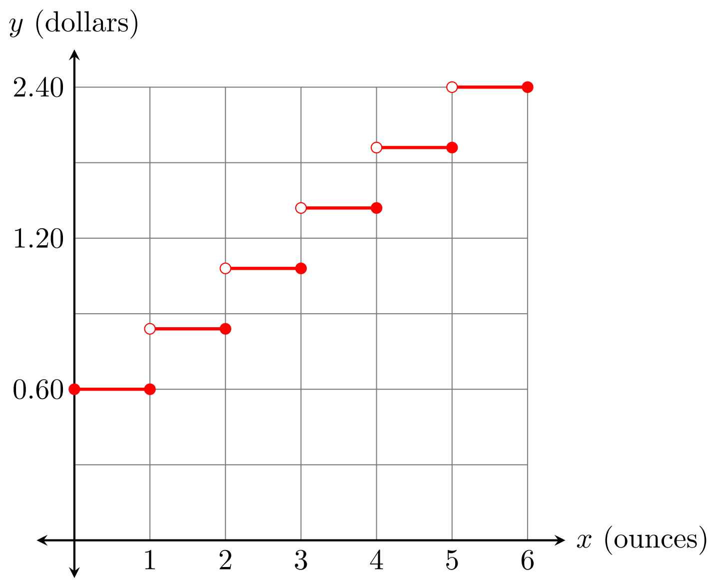


Figure 6.2: Graph of 2022 postal rates.

Notice the jumps. If your letter is exactly one ounce, you pay \$0.60 to mail it. But if it's the slightest bit over, you pay \$0.84. There is no letter which will ever cost any other price between \$0.60 and \$0.84.

How do we describe this using limits? We will introduce the concepts of **left-handed limits** and **right-handed limits.**

In Figure 6.3, we have zoomed in on a part of the graph near $x = 1$. Notice there is a thin strip blocking out the part of the graph right at $x = 1$. As we saw in calculating derivatives, you can't look at $h = 0$ right away, since it's undefined. So in looking at limits, we look *around* an x-value and see what's happening.

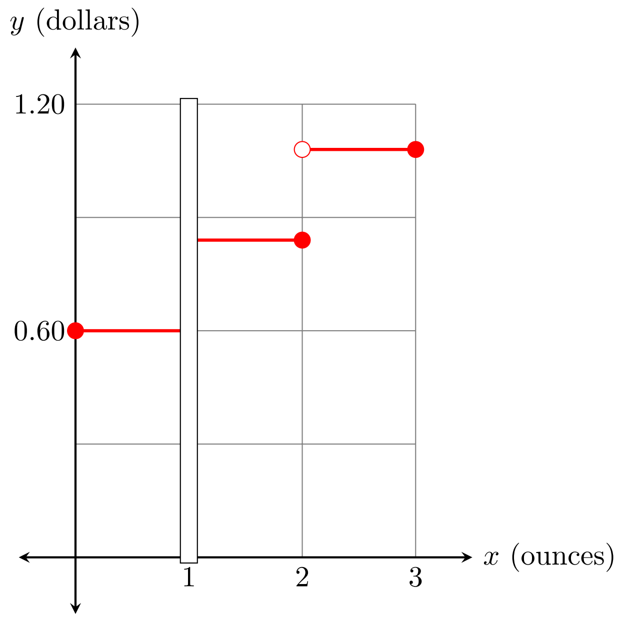


Figure 6.3: Graph of 2022 postal rates (closeup).

Start by looking at what happens when x moves to 1 coming from the left. It looks like at $x = 1$, the value of the function is 0.60. But if we start a little bit to the right of $x = 1$ and move *left,* it looks like the value of the function is 0.84. (Of course it can't be *both,* since one input, $x = 1$, cannot have two outputs.) In the language of limits, we say

$$\lim_{x \to 1^-} f(x) = 0.60, \qquad \lim_{x \to 1^+} f(x) = 0.84.$$

We read these as "the limit as x approaches 1 from the left of $f(x)$ is 0.60," and "the limit as x approaches 1 from the right of $f(x)$ is 0.84." In general, the "$-$" as a superscript means looking from the left, and the "$+$" means looking from the right.

Now we know that $f(1) = 0.60$ from Figure 6.2. So

$$\lim_{x \to 1^-} f(x) = 0.60, \quad f(1) = 0.60, \quad \lim_{x \to 1^+} f(x) = 0.84.$$

This is how, using the language of limits, we can say that there is a jump in a graph.

Example 2

Let's take a look at another example with a jump in the graph.

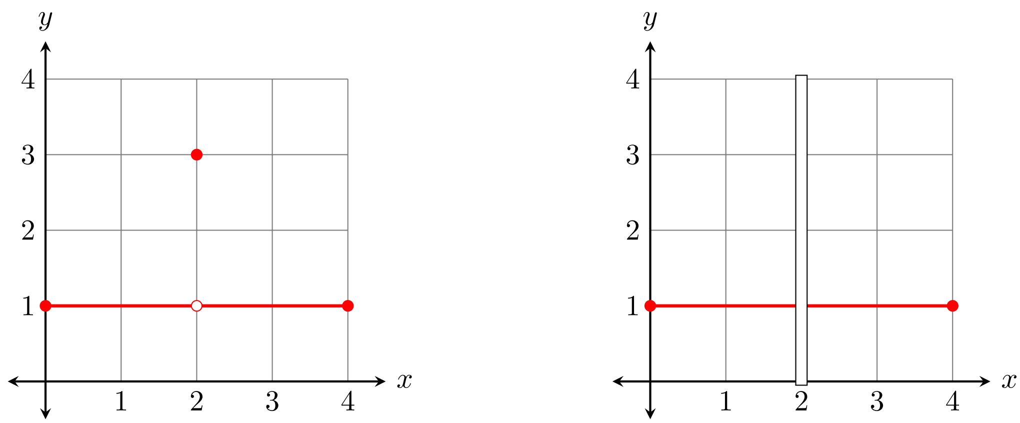


Figure 6.4: Graph of a function $g(x)$ with a removable discontinuity.

What is happening at $x = 2$? Looking at the right graph in Figure 6.4, we can see that

$$\lim_{x\to 2^-} g(x) = 1 = \lim_{x\to 2^+} g(x).$$

When both left-hand and right-hand limits are the same, we can simply say

$$\lim_{x\to 2} g(x) = 1.$$

But $g(2) = 3 \neq 1$, so there is a discontinuity at $x = 2$. We say this type of discontinuity is a **removable discontinuity** since we can redefine $g(x)$ at 2 to make it continuous at $x = 2$. If we make $g(2) = 1$, the function would be continuous at $x = 2$.

Note the difference at $x = 1$ in Figure 6.3. It doesn't matter how we define $f(1)$, there *must* be a jump. In this case, we say that $x = 1$ is an **essential discontinuity.**

Example 3

Let's take a look an example without a jump. In this case, $h(x) = x + 1$.

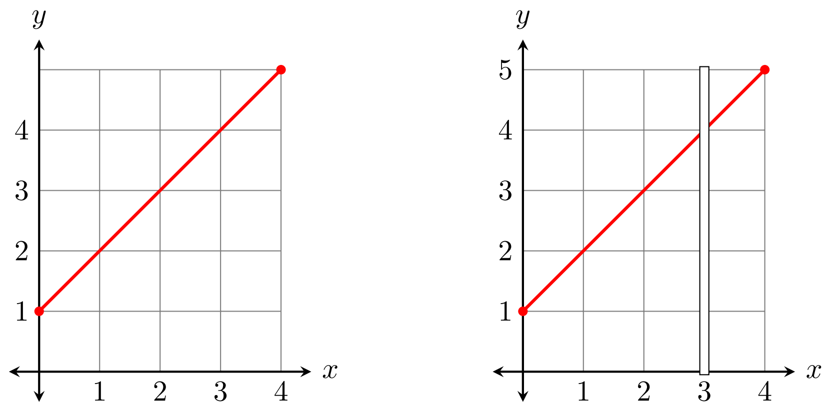


Figure 6.5: Graph of a function $h(x)$ without a jump.

Let's look at what is happening at $x = 3$. Looking at the right graph in Figure 6.5, we can see that

$$\lim_{x \to 3^-} h(x) = 4 = \lim_{x \to 3^+} h(x).$$

Since both left-hand and right-hand limits are the same, we can say

$$\lim_{x \to 3} h(x) = 4.$$

But $h(3) = 4$ as well, so we can say that $h(x)$ is **continuous** at $x = 3$.

If a function is continuous at *every* point in its domain, we say **the function is continuous.** If we are only looking at a single point, we just say that a function is continuous at a particular point, as in this example.

Here is a summary of the new terminology we encountered in the first three examples.

"$\lim_{x \to a^-} f(x)$" means the limit of $f(x)$ as x approaches a from the left.

"$\lim_{x \to a^+} f(x)$" means the limit of $f(x)$ as x approaches a from the right.

Let $f(x)$ be given and let a be in its domain.

If...	then...
$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+}$	$f(x)$ has an essential discontinuity at a
$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} \neq f(a)$	$f(x)$ has a removable discontinuity at a
$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} = f(a)$	$f(x)$ is continuous at a

It is important to note that the limits mentioned in this chart do not always exist for every function. We will look at such examples later; for now, we'll stick to functions where these limits exist.

Example 4

Most of the examples we'll come across will be continuous. Proving that a function is continuous can often involve a lot of work with limits. To avoid all these proofs, we'll summarize the main points below.

The following functions are continuous wherever they are defined:

1. Polynomials, such as $f(x) = 3x^5 - 4x^2 + 7$,
2. Roots/radicals, such as $f(x) = x^{2/3}$,
3. Rational functions, such as $f(x) = \dfrac{3x^2 - 1}{x + 4}$,
4. The basic trigonometric functions: $\sin(x)$, $\cos(x)$, and $\tan(x)$,
5. $f(x) = e^x$,
6. $f(x) = \ln x$.

If $f(x)$ and $g(x)$ are continuous functions, the following are also continuous:

7. $c \cdot f(x)$, where c is a constant,
8. $f(x) + g(x)$,
9. $f(x) - g(x)$,
10. $\dfrac{f(x)}{g(x)}$, as long as $g(x) \neq 0$,
11. $(f \circ g)(x)$.

What this means is that it's very easy to create new continuous functions from the basic ones. To see an example, let's show that $h(x) = \sin(x^2 + 1)$ is a continuous function. First, we write $h(x)$ as a function composition. With $f(x) = \sin(x)$ and $g(x) = x^2 + 1$, we have $h(x) = (f \circ g)(x)$. But $f(x)$ is continuous by (4) above and $g(x)$ is continuous by (1) above. Then by (11), their composition is continuous, which is just $h(x)$.

Example 5

Let's look now at $f(x) = \dfrac{1}{x}$. Note that here, the domain is all real numbers *except* $x = 0$. By (3) above, we know that $f(x)$ is a continuous function. But how does that make sense? Doesn't the graph make a huge jump across $x = 0$?

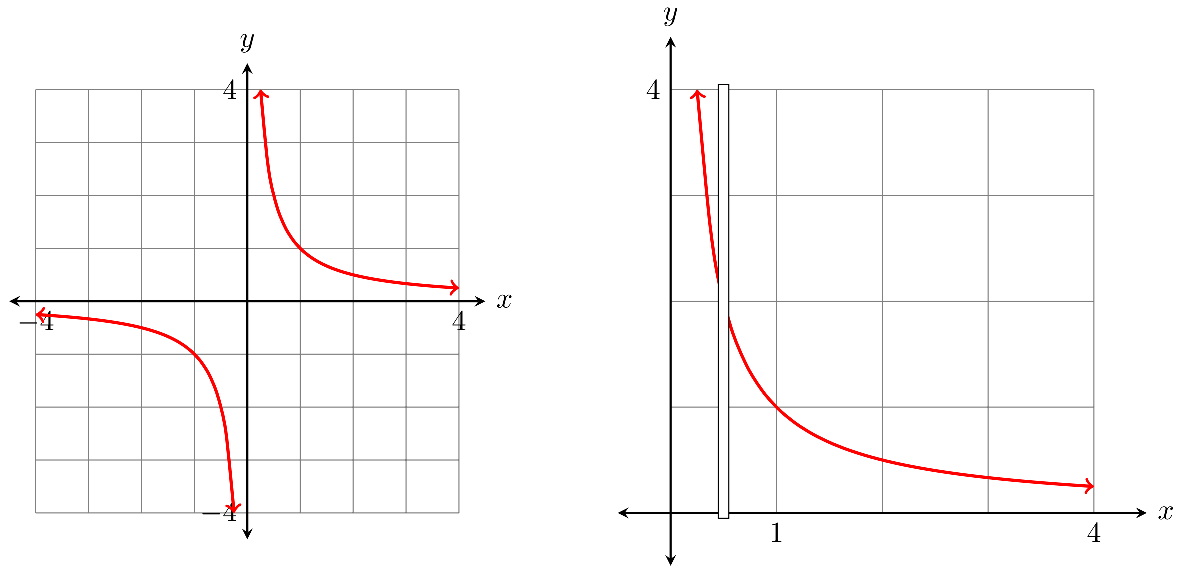


Figure 6.6: Graph of $f(x) = \dfrac{1}{x}$ (left), closeup at $x = \dfrac{1}{2}$ (right).

The important concept here is that you can only talk about continuity where a function is *defined.* Look at the right graph in Figure 7.1. Is $f(x)$ continuous at $x = \dfrac{1}{2}$? You should convince yourself that

$$\begin{aligned}\lim_{x\to 1/2^-} f(x) &= \lim_{x\to 1/2^+} f(x)\\ &= f\left(\frac{1}{2}\right)\\ &= 2.\end{aligned}$$

This means precisely that $f(x)$ is continuous at $\dfrac{1}{2}$. We can draw a similar thin vertical strip and make similar observations at any point on the graph.

But we *cannot* draw a vertical strip at $x = 0$, since $f(0)$ is *undefined.* This seems like we're being a bit picky, but this concept is *very* important to understanding continuity. It will be especially important when we explore asymptotes of functions.

Example 6

Consider the graph of $f(x)$ shown In Figure 6.7. Describe the behavior at $x = 0$, $x = 1$, and $x = 2$ using what we have just learned.

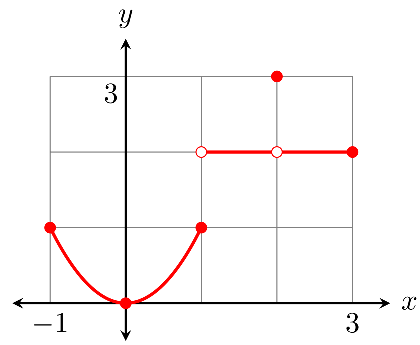


Figure 6.7: Describing continuity of $f(x)$.

At $x = 0$, we have $\lim_{x\to 0^-} f(x) = 0$ and also $\lim_{x\to 0^+} f(x) = 0$. Since $f(0) = 0$ as well, $f(x)$ is continuous at $x = 0$.

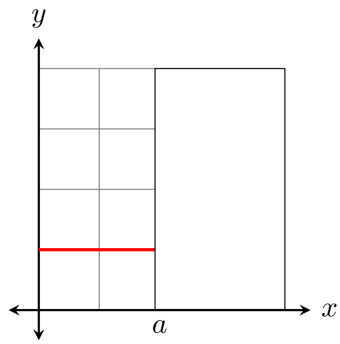
At $x = 1$, we have $\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = 2$. Since these limits are not equal, there is an essential discontinuity at $x = 1$.

At $x = 2$, we see that $\lim_{x\to 2^-} f(x) = 2$ and also $\lim_{x\to 2^+} f(x) = 2$ as well. But $f(2) = 3$, which is *not* equal to 2. Therefore, there is a removable discontinuity at $x = 2$.

Summary

The limit of $f(x)$ as x approaches a from the left:

$$\lim_{x \to a^-} f(x)$$

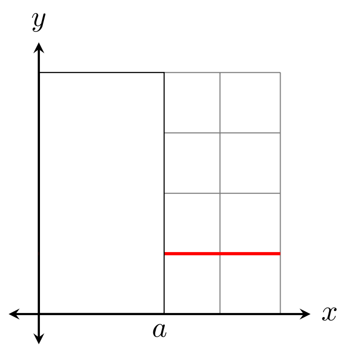


Used for:

1. Determining continuity,
2. Describing discontinuities,

The limit of $f(x)$ as x approaches a from the right:

$$\lim_{x \to a^+} f(x)$$



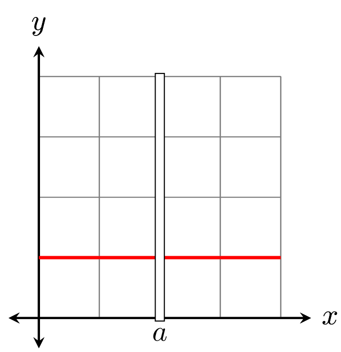
Used for:

1. Determining continuity,
2. Describing discontinuities,

The limit of $f(x)$ as x approaches a : $\lim_{x \to a} f(x)$.

Only exists if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$



Used for:

1. Determining continuity,
2. Describing discontinuities.

Homework

1. Suppose a function $f(x)$ is defined for all real numbers. You know that

$$\lim_{x\to 2^-} f(x) = 3, \quad \lim_{x\to 2^+} f(x) = 4.$$

   Which of the following are possible? Circle all that apply.

   (a) $f(x)$ is continuous at $x = 2$.
   (b) $f(x)$ has an essential discontinuity at $x = 2$.
   (c) $f(x)$ has a removable discontinuity at $x = 2$.

2. Suppose a function $f(x)$ is defined for all real numbers. You know that

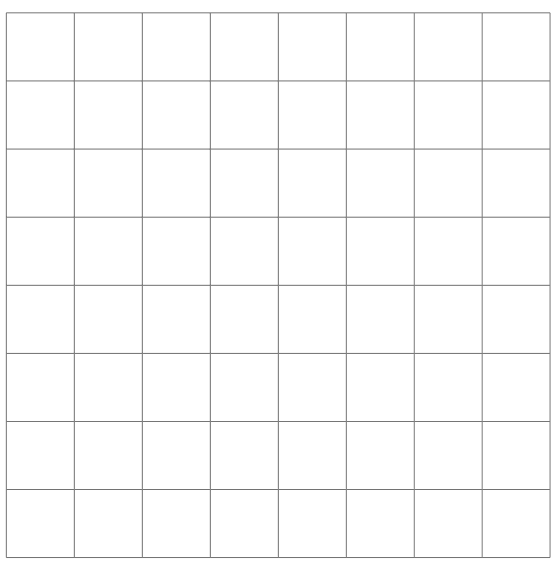
$$\lim_{x\to -1^+} f(x) = f(-1).$$

   Which of the following are possible? Circle all that apply.

   (a) $f(x)$ is continuous at $x = -1$.
   (b) $f(x)$ has an essential discontinuity at $x = -1$.
   (c) $f(x)$ has a removable discontinuity at $x = -1$.

3. On the grid below, sketch a graph of a function $f(x)$ which has the following properties. Note: many answers are possible; there is not just one correct answer.

   (a) There is a removable discontinuity at $x = 1$.
   (b) There is an essential discontinuity at $x = 3$.
   (c) $\lim_{x\to -1^-} f(x) = -2$.
   (d) $\lim_{x\to 3^+} f(x) = 2$.
   (e) $f(-2) = -1$.
   (f) $f(x)$ is continuous at $x = -2$.



4. Suppose $f(x) = \dfrac{1}{x^2 - 1}$. The graph is shown below.

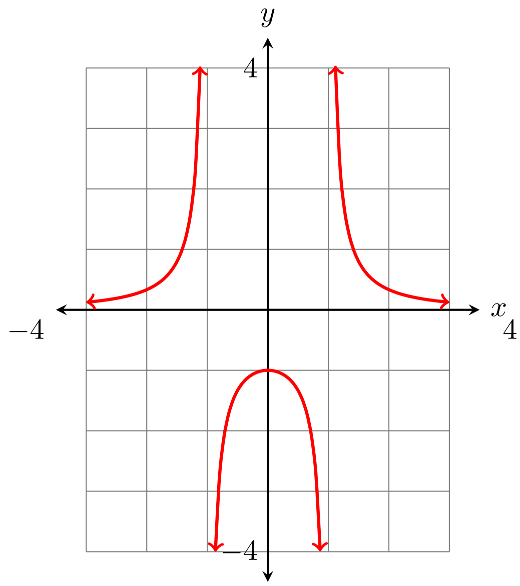


Figure 6.8: Graph of $f(x) = \dfrac{1}{x^2 - 1}$.

You are discussing with your friend whether or not this is a continuous function. She says, "No way! Look at how the function jumps around! It looks like there are vertical asymptotes. This function can't be continuous!"

What is your repsonse to your friend?

5. The function $f(x) = \lceil x \rceil$ is called the *ceiling function.* It is defined so that $f(x)$ is the smallest integer that is greater than or equal to x. Thus,

$$\lceil 4 \rceil = 4, \quad \lceil 4.5 \rceil = 5, \quad \lceil -2 \rceil = -2, \quad \lceil -1.5 \rceil = -1.$$

Part of its graph is shown below.

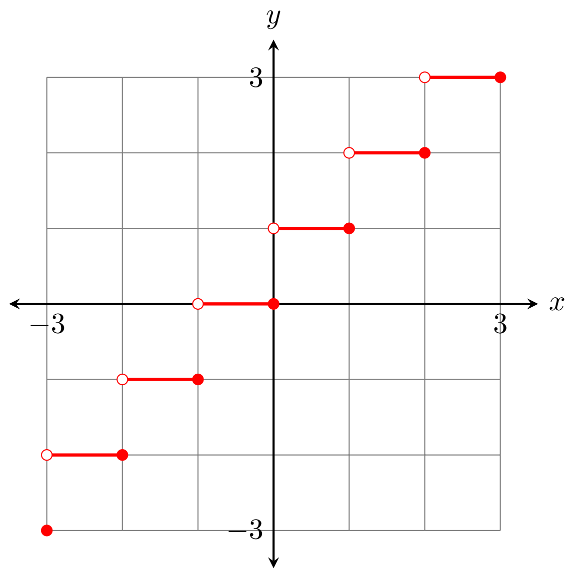


Figure 6.9: Partial graph of the ceiling function.

(a) What is $f(-4.8)$? $f(4.8)$?

(b) Describe the behavior of the graph at $x = 0$ using the notations and terminology of this section.

(c) Choose the best answer. At $x = 0.5$, the function:

i. Is continuous.

ii. Has an essential discontinuity.

iii. Has a removable discontinuity.

6. Suppose you are given the following graph, with a thin vertical strip covering $x = a$. Which of the following are possible? Circle all that apply.

   (a) $f(x)$ is continuous at $x = a$.

   (b) $f(x)$ has an essential discontinuity at $x = a$.

   (c) $f(x)$ has a removable discontinuity at $x = a$.

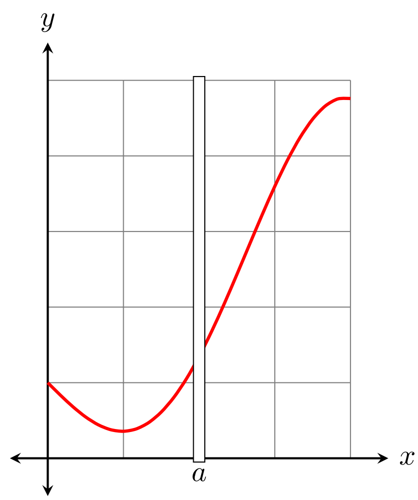


Figure 6.10: Graph of a function $f(x)$.

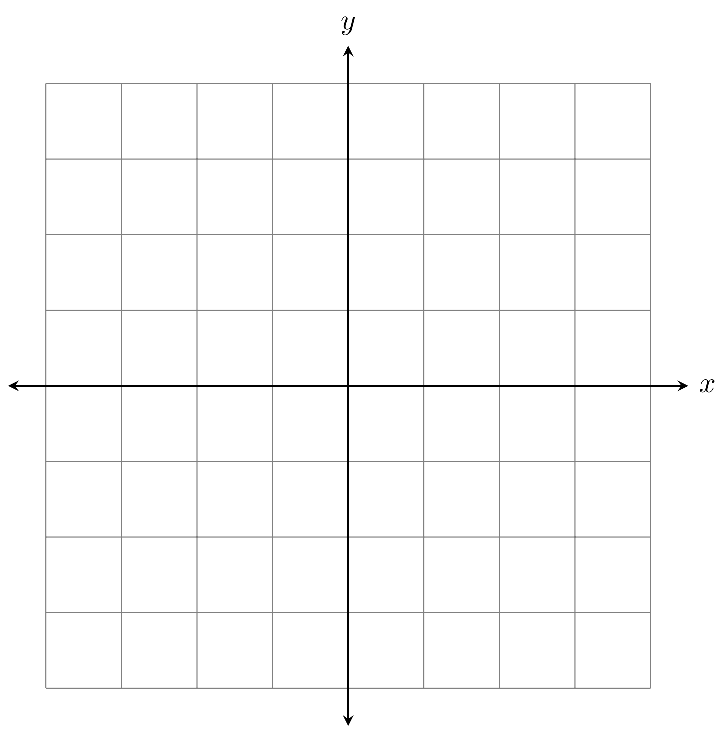
7. Sometimes it makes more than one formula to define a function. You might have seen the absolute value function, $f(x) = |x|$, defined in a **piecewise** way:

$$f(x) = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Consider the following piecewise-defined function. Assume b is a constant.

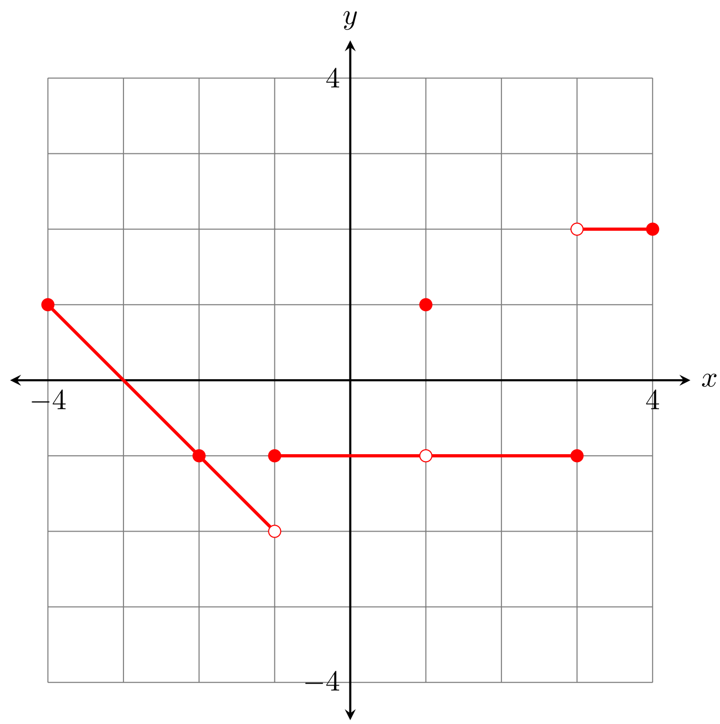
$$g(x) = \begin{cases} x+1, & x \leq 1, \\ b-x, & x > 1. \end{cases}$$

What must be the value of b so that $g(x)$ is a continuous function? Sketch a graph of this function on the interval $[-4, 4]$ below.



Solutions

1. (b).
2. (a) and (b).
3. Many other answers are possible. This is just one way to satisfy all the given properties.



4. You would tell your friend to be very careful with the definition of continuity! Continuity can only be discussed where a function is *defined.* Since the function is *not* defined at $x = -1, 1$, these values cannot detract from the continuity of the function. Wherever the function *is* defined, it is continuous. (See Example 5 for another example.)
5. (a) $f(-4.8) = -4$, $f(4.8) = 5$.

   (b)
$$\lim_{x \to 0^-} f(x) = 0, \qquad \lim_{x \to 0^+} f(x) = 1.$$
   Therefore, there is an essential discontinuity at $x = 0$.

   (c) (i).
6. (a), (c).

7. $g(x)$ has pieces which are lines (which are polynomials), so we have to check that they "meet up" at $x = 1$. Keeping in mind that b is a constant, we have

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} (x+1) = 2$$

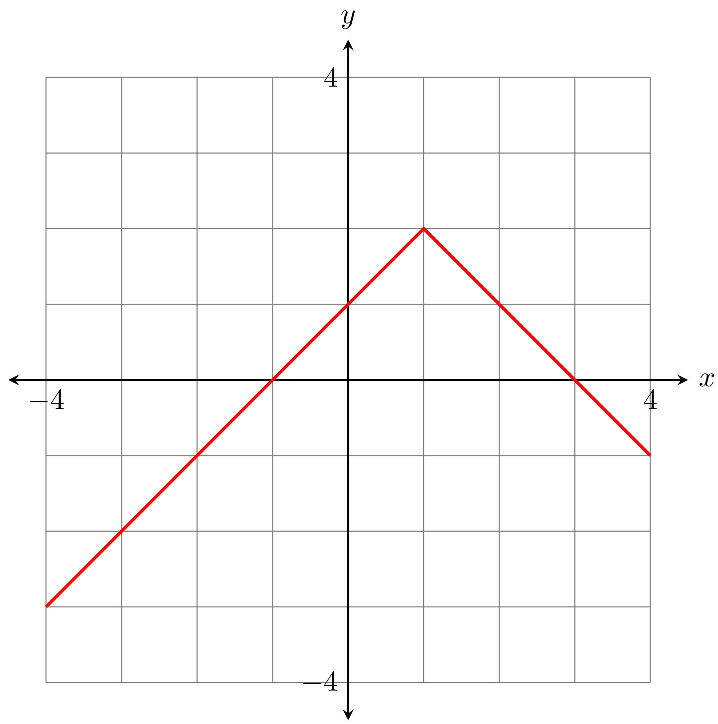
and

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (b-x) = b-1.$$

To be continuous at $x = 1$, these left-hand and right-hand limits must be equal, and so

$$2 = b - 1.$$

This gives $b = 3$. We graph the function $x + 1$ to the left of $x = 1$ and the function $3 - x$ to the right of $x = 1$. We chose b so that they meet up perfectly at $x = 1$.



6.2 Optimization

In calculus, the term **optimization** involves finding mimima or maxima of a function. For example, when is the tide the highest? What price will maximize profit? How can you build a box using the least amount of wood? These are questions of optimization.

First, we need a little terminology. We've used the terms "minimum" and "maximum" informally, but now we need to be a little more precise. Let's look at the graph in Figure 6.11.

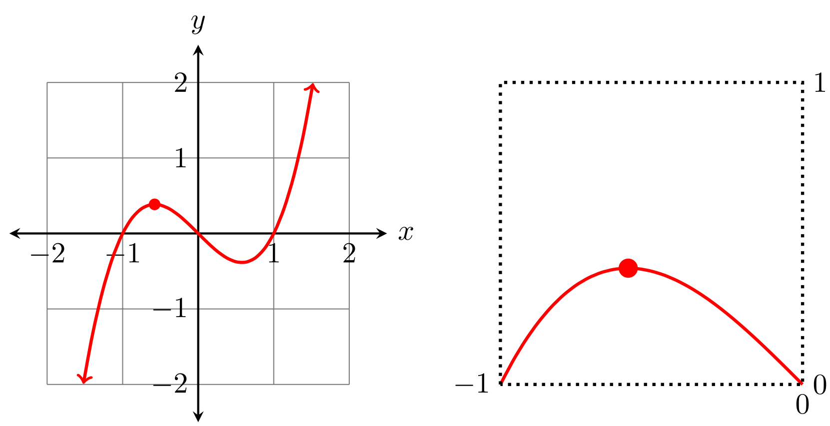


Figure 6.11: Local extrema: zooming in on $f(x) = x^3 - x$.

Near -0.6, the graph has a **local maximum** (plural **local maxima**). In other words, if we zoom in closer, as on the right of Figure 6.11, it looks like the highest point on the graph is near $x = -0.6$. If we zoom back out (the left graph of Figure 6.11), we see that this point is not the highest point on *the entire graph.* So it is not a **global maximum** (plural **global maxima**) – the highest point on a graph.

Near $x = 0.6$, we see that these is a **local minimum** (plural **local mimima**) – if we zoom in, it will look like the graph has a lowest point at $x \approx 0.6$. But it is not a **global minimum** (plural **global minima**), since it is not the lowest point on *the entire graph.*

We use the term **local extremum** (plural **local extrema**) to mean either a local minimum or maximum, and the term **global extremum** (plural **global extrema**) to mean either a global minimum or global maximum. Another common term for global extremum is **absolute extremum.** You will likely see both.

Looking for...	think...
Local extrema	Zooming in
Global extrema	Zooming out

Example 1

Consider the graph of $f(x)$ shown below. The arrows means that the graph keeps going up (it's actually a fourth-degree polynomial).

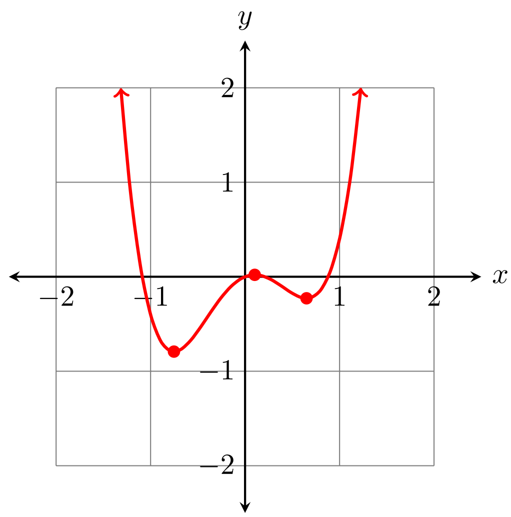


Figure 6.12: Local and global extrema on the graph of $f(x) = 2x^4 - 2x^2 + 0.4x$.

Let's look at some features of this graph using our new terminology. At $x \approx -0.8$, we have a local minimum *and* a global minimum, as this point is the lowest on the entire graph. At $x \approx 0.7$, we have a local minimum – but it's not a global minimum because there are lower points on the graph. At $x \approx 0.1$, we have a local maximum – but it's not a global maximum since the graphs extends upward toward infinity. There is no global maximum on this graph.

It is worth pointing out that some graphs have *no* local or global extrema. Take the exponential function $f(x) = e^x$, for example (see below). It is always increasing, so there can be no local or global maxima. There is no local or global minimum, either. You might be tempted to think that 0 is a global minimum. But it is not possible to solve $e^x = 0$, so there is *no* x-value that has a y-value of 0. So there are no minima, either.

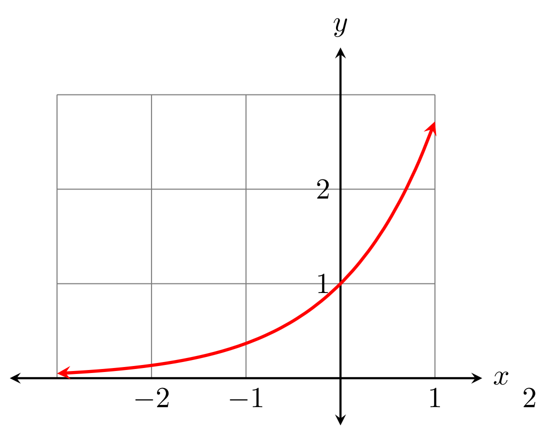


Figure 6.13: Graph of $f(x) = e^x$.

Example 2

We now look at how to find extrema of a function. We consider the function $f(x) = \sin(x)$ on the interval $[0, 4\pi]$. Note that when we restrict the domain (the x-values), we are looking for global extrema over the interval $[0, 4\pi]$ *only*. This is very common in mathematics and science. Often, the horizontal axis represents time, and you only ever consider some finite period of time, not an infinite period.

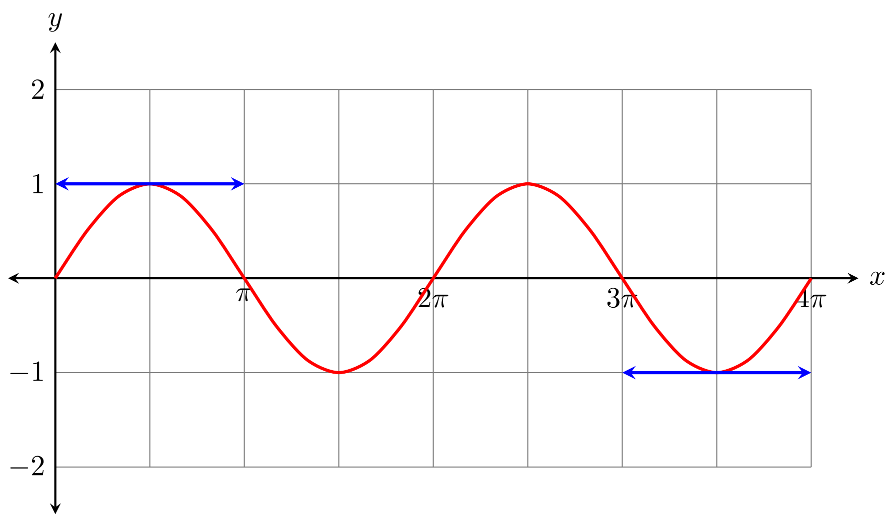


Figure 6.14: Graph of $f(x) = \sin(x)$ on the interval $[0, 4\pi]$.

We observed earlier that when we have a local extremum, we have a horizontal tangent, as seen in Figure 9.17. But a horizontal line has a slope of 0, and so 0 is the slope of the tangent line – which is given by the derivative. So our strategy should be to find out where $f'(x) = 0$.

But $f'(x) = \cos(x)$. From the unit circle, we know that solving $\cos(x) = 0$ on the interval $[0, 4\pi]$ gives four solutions:

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}.$$

We know what type of extrema these are by looking at the graph. Can we do this *without* a graph? The key observation is that at a local maximum, the graph is concave down, and at a local minimum, the graph is concave up. We determine whether a graph is concave down or up by looking at $f''(x)$. So

$$f'(x) = \cos(x)$$
$$f''(x) = -\sin(x)$$

So

$$f''\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1 < 0,$$

so the graph must be concave down at $x = \dfrac{\pi}{2}$ since the second derivative is negative there, giving a local maximum. Similarly,

$$f''\left(\frac{3\pi}{2}\right) = -\sin\left(\frac{3\pi}{2}\right) = 1 > 0,$$

so the graph must be concave up at $x = \dfrac{3\pi}{2}$ since the second derivative is positive there, giving a local minimum. There is also a local maximum at $x = \dfrac{5\pi}{2}$ and a local minimum at $x = \dfrac{7\pi}{2}$.

Are there any global extrema? It turns out there are *two* global extrema, $\left(\dfrac{\pi}{2}, 1\right)$ and $\left(\dfrac{5\pi}{2}, 1\right)$, since there are two values of x where $f(x) = 1$ (again, restricting attention to the given domain). Likewise, there are two global minima at $x = \dfrac{3\pi}{2}$ and $x = \dfrac{7\pi}{2}$.

Two important points to take away: there may be multiple global extrema, and we can use the second derivative to help us determine if local extrema are minima or maxima.

Example 3

In this example, we consider the function $f(x) = 1 - x^4$, shown in Figure 6.15.

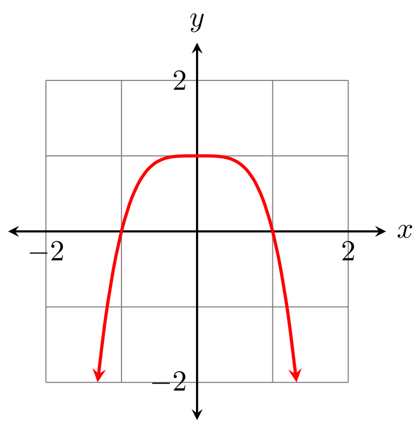


Figure 6.15: Graph of $f(x) = 1 - x^4$.

We can observe a local and global maximum at the point $(0, 1)$. Let's use calculus to verify this. Remember, there is a horizontal tangent there, we need to find out where $f'(x) = 0$.

$$\begin{aligned} f(x) &= 1 - x^4 \\ f'(x) &= -4x^3 \\ -4x^3 &= 0 \\ x &= 0 \end{aligned}$$

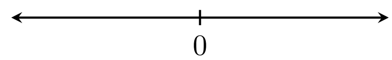
So, as expected, we have $f'(0) = 0$. Now let's try the second derivative:

$$\begin{aligned} f'(x) &= -4x^3 \\ f''(x) &= -12x^2 \\ f''(0) &= 0 \end{aligned}$$

So since $f''(0) = 0$, we can't tell whether the graph is concave up or concave down there – there might even be an inflection point. So we need to use a sign chart.

1. We already know that solving $f''(x) = 0$ gives $x = 0$.

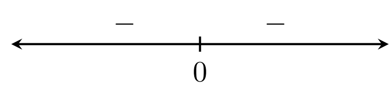
2. This gives the following number line:



3. Now choose one value from each interval. Easy values are $x = -1$ and $x = 1$.

$$\begin{aligned} f''(-1) &= -12(-1)^2 \\ &= -12 \\ &< 0 \\ f''(1) &= -12(1^2) \\ &- 12 \\ &< 0. \end{aligned}$$

   This yields the following number line:



   There is no inflection point since the concavity does not change – it's concave down on both sides of $x = 0$. This means that $x = 0$ is a local maximum, and in this case, also a global maximum.

This and the last example show you that to find local extrema, we set $f'(x) = 0$. To see if the function is concave up or down, use $f''(x)$. This works *except* when $f''(x) = 0$, in which case you need to make a sign chart. Here's a summary.

To find local extrema of $f(x)$:

1. Determine where $f'(x) = 0$.
2. Find $f''(x)$ at these points.
   (a) If $f''(x) > 0$, there is a local minimum.
   (b) If $f''(x) < 0$, there is a local maximum.
   (c) If $f''(x) = 0$, use a sign chart for $f''(x)$.

Global Minima and Maxima

Not every function has local or global extrema. But in certain circumstances, we can know that *global* extrema do in fact exist.

> **Extreme Value Theorem**
> If a function is defined on a closed interval and is continuous, both a global minimum and a global maximum exist.

What is so important about a closed interval? Let's look at $f(x) = \dfrac{1}{x}$, shown in Figure 8.13.

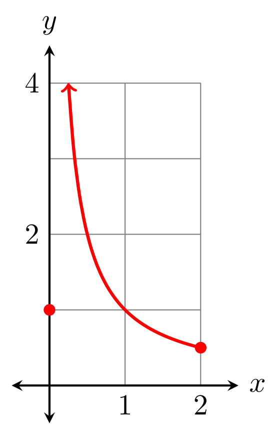


Figure 6.16: The importance of a closed interval.

We know what the graph looks like on $(0, 2]$; there is a vertical asymptote. Now suppose we wanted to create a continuous function on $[0, 2]$ by defining the function to be some value at $x = 0$. Can you see why this is impossible? No matter how we defined $f(0)$ – for example, $f(0) = 1$ – in order to be continuous at 0, the function would somehow have to turn around and come back down to the point $(0, 1)$. This cannot be done if there is a vertical asymptote at $x = 0$.

Essentially, by making the assumption that the function is defined on a closed interval, it is not possible for there to be any vertical asymptotes. So there must be a lowest and highest point somewhere on the graph. A formal proof is a bit more complicated, but the graph in Figure 8.13 is meant to give you an idea of why this must be true.

So if we know that global extrema exist, how do we find them? There is a straightforward way using calculus. First, we'll give the method and then do some examples.

Suppose a function $f(x)$ is defined on a closed interval $[a, b]$ and is continuous. Then both a global minimum and a global maximum exist. To find them:

1. Determine where $f'(x) = 0$ or $f'(x)$ does not exist,
2. Evaluate $f(x)$ at these points and the endpoints a and b,
3. Select the lowest and highest values among these function values.

Example 4

Let the function $f(x) = 3x - x^3$ be defined on the closed interval $[-2, \sqrt{3}]$. Find the global extrema. The graph is shown in Figure 6.17.

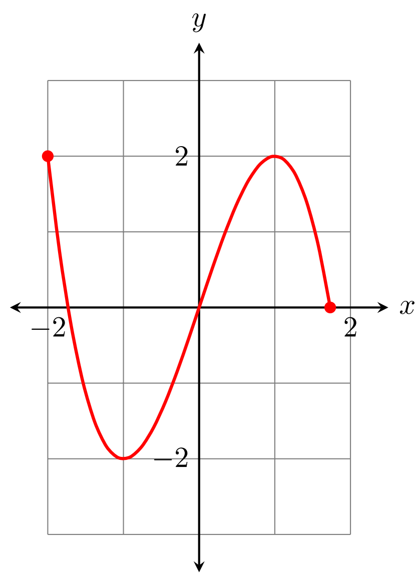


Figure 6.17: The graph of $f(x) = 3x - x^3$ on the closed interval $[-2, \sqrt{3}]$.

Let's proceed with the steps one by one.

1. Using the Power Rule, we get $f'(x) = 3 - 3x^2$. Since the derivative is a polynomial, it exists everywhere. To see where it's 0, we solve.

$$\begin{aligned} f'(x) &= 0 \\ 3 - 3x^2 &= 0 \\ 3 &= 3x^2 \\ x^2 &= 1 \\ x &= -1, +1 \end{aligned}$$

2. Now evaluate at these points and the endpoints. Note that we want function values here, so we plug into $f(x)$.

$$f(-1) = -2, \quad f(1) = 2, \quad f(-2) = 2, \quad f(\sqrt{3}) = 0.$$

3. Looking at these function values, -2 is the lowest and 2 is the highest. Thus, there is a global minimum at $(-1, -2)$, and global maxima at $(-2, 2)$ and $(1, 2)$. Of course, these results make perfect sense by looking at the graph.

Assessment Expectations: When finding extrema, you will always be given a graph. But you must find the extrema using calculus, and use the graph to verify your results. You will not be given partial credit for just using the graph.

Example 5

It's important to note that you do *not* have to assume that the derivative exists everywhere in order to find global extrema. Let's take the example of $f(x) = |x|$ on the closed interval $[-2, 1]$, shown in Figure 9.13.

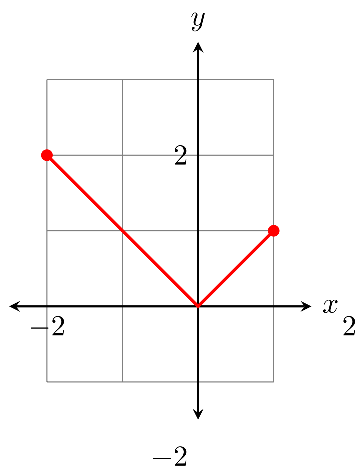


Figure 6.18: The graph of $f(x) = |x|$ on the closed interval $[-2, 1]$.

Now let's find the global extrema.

1. What is $f'(x)$? When $x < 0$, the graph is a line with slope -1, and so $f'(x) = -1$ in this case. When $x > 0$, the graph is a line with slope 1, and so $f'(x) = 1$ here. But what happens when $x = 0$?

   First, remember the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

   We can use this definition to look at a specific value of x, in this case, $x = 0$. So we'll substitute $x = 0$ and see what happens.

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{|0+h| - |0|}{h} \\ &= \lim_{h \to 0} \frac{|h|}{h}. \end{aligned}$$

   You can't cancel the h out of an absolute value, so we need to make a chart.

h	$\lvert h\rvert/h$
-0.01	-1
-0.001	-1
-0.0001	-1
0.01	1
0.001	1
0.0001	1

Table 6.1: Approximating $\lim_{h\to 0} \frac{|h|}{h}$.

You can see that the limit does not exist, since $-1 \neq 1$. This means you *cannot* take the derivative at $x = 0$, and so $f'(0)$ does not exist. You can see there is a sharp corner at $x = 0$ on the graph; this is why a differentiable function is sometimes called a **smooth** function. And since $f(x)$ is not differentiable at 0, we have to include $x = 0$ when checking for global extrema.

2. Evaluating the function:
$$f(0) = 0, \quad f(-2) = 2, \quad f(1) = 1.$$

3. The lowest function value is 0, so there is a global minimum at $(0, 0)$. The highest function value is 2, so there is a global maximum at $(-2, 2)$.

Example 6

Let's look at another example where we need to look at where $f'(x)$ doesn't exist. Consider $f(x) = \sqrt{x}$ on the closed interval $[0, 4]$, shown in Figure 6.19.

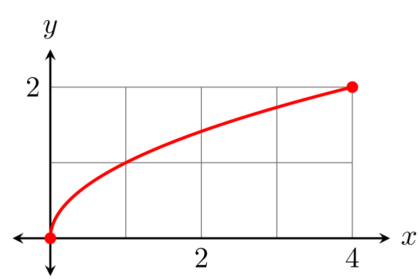


Figure 6.19: The graph of $f(x) = \sqrt{x}$ on the closed interval $[0, 4]$.

Again, let's apply the steps.

1. Writing $f(x) = x^{1/2}$, we use the Power Rule to get
$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

Note that the derivative can never be 0, because the numerator is always 1. But there *is* a place the derivative is undefined: $x = 0$. This is because you can't have a 0 in the denominator. What this means on the graph is that there is a vertical tangent at $x = 0$, and we know that vertical lines have an undefined slope.

2. Evaluating the function:
$$f(0) = 0, \quad f(4) = 2.$$

3. Looking for lowest and highest values, we have a global minmum at $(0, 0)$ and a global maximum at $(4, 2)$. It turns out that the only place $f'(x)$ is undefined is at an endpoint, but that won't always be the case.

Homework

1. Find the local extrema for the function $f(x) = x - 2\cos(x)$ on the interval $[0, 2\pi]$. Check that you're right by graphing.
2. Find the local extrema for the function $f(x) = x^5$ using the method in the notes. We can look at a graph and see that there are none, but use calculus to show it.
3. Find the global extrema for the function $f(x) = x^{2/3}$ on the closed interval $[-4, 4]$. Graph this function on desmos (or your calculator) to verify your answer.
4. Find the global extrema for the function $f(x) = e^x - x$ on the closed interval $[-5, 2]$.
5. Find the global extrema for the function $f(x) = x - \dfrac{1}{4}\ln x$ on the closed interval $[1, 7]$.

Solutions

1. (a) We first find where $f'(x) = 0$ using the unit circle.

$$\begin{aligned} f(x) &= x - 2\cos(x) \\ f'(x) &= 1 + 2\sin(x) = 0 \\ \sin(x) &= -\frac{1}{2} \\ x &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

(b) Next, we evaluate $f''(x)$ at these points.

$$\begin{aligned} f'(x) &= 1 + 2\sin(x) \\ f''(x) &= 2\cos(x) \end{aligned}$$

Since $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2} < 0$, there is a local maximum at $x = \frac{7\pi}{6}$. Since $\cos\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2} > 0$, there is a local minimum at $x = \frac{7\pi}{6}$.

2. (a) We first find where $f'(x) = 0$.

$$\begin{aligned} f(x) &= x^5 \\ f'(x) &= 5x^4 \\ 5x^4 &= 0 \\ x &= 0 \end{aligned}$$

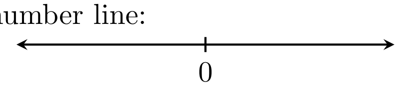
(b) Now check $f''(x)$.

$$\begin{aligned} f'(x) &= 5x^4 \\ f''(x) &= 20x^3 \\ f''(0) &= 0 \end{aligned}$$

Since $f''(0) = 0$, we need to make a sign chart.

i. We already know that solving $f''(x) = 0$ gives $x = 0$.

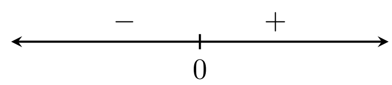
ii. This gives the following number line:



iii. Now choose one value from each interval. Easy values are $x = -1$ and $x = 1$.

$$\begin{aligned} f''(-1) &= 20(-1)^3 \\ &= -20 \\ &< 0 \\ f''(1) &= 20(1)^3 \\ &20 \\ &> 0. \end{aligned}$$

This yields the following number line:



Thus, since the concavity changes at $x = 0$, there must be an inflection point there, and therefore no local extrema exist.

3. (a) First, we determine when $f'(x) = 0$ or when $f'(x)$ does not exist.

$$\begin{aligned} f(x) &= x^{2/3} \\ f'(x) &= \frac{2}{3}x^{-1/3} \\ &= \frac{2}{3\sqrt[3]{x}} \end{aligned}$$

$f'(x)$ can never be 0 since the numerator cannot be 0. $f'(x)$ is undefined when $x = 0$ since you can't have 0 on the denominator.

(b) Now evaluate at this point and the endpoints.

$$f(0) = 0, \quad f(-4) = \sqrt[3]{16}, \quad f(4) = \sqrt[3]{16}.$$

(c) Looking at largest and smallest values, we have a global minimum at $(0, 0)$ and global maxima at $(-4, \sqrt[3]{16})$ and $(4, \sqrt[3]{16})$.

4. (a) First, we determine when $f'(x) = 0$ or when $f'(x)$ does not exist.

$$\begin{aligned} f(x) &= e^x - x \\ f'(x) &= e^x - 1 = 0 \\ e^x &= 1 \\ x &= 0 \end{aligned}$$

$f'(x)$ always exists since e^x exists for every x.

(b) Now evaluate at this point and the endpoints.

$$f(0) = e^0 - 1 = 0, \quad f(-5) = e^{-5} - (-5) \approx 4.99, \quad f(2) = e^2 - 2 \approx 5.39.$$

(c) Looking at largest and smallest values, we have a global minimum at $(0, 0)$ and global maximum at $(2, e^2 - 2)$.

5. (a) First, we determine when $f'(x) = 0$ or when $f'(x)$ does not exist.

$$\begin{aligned} f(x) &= x - \frac{1}{4}\ln x \\ f'(x) &= 1 - \frac{1}{4} \cdot \frac{1}{x} = 1 - \frac{1}{4x} = 0 \\ \frac{1}{4x} &= 1 \\ 4x &= 1 \\ x &= \frac{1}{4} \end{aligned}$$

However, we cannot consider this point since $\frac{1}{4}$ is *not* in the interval $[1, 7]$.

$f'(x)$ always exists since the denominator cannot be 0 since we are looking at the closed interval $[1, 7]$.

(b) Now evaluate at the endpoints.

$$f(1) = 1 - \frac{1}{4}\ln 1 = 1, \quad f(7) = 7 - \frac{1}{4}\ln 7 \approx 6.51$$

(c) Looking at largest and smallest values, we have a global minimum at $(1, 1)$ and global maximum at $\left(7, 7 - \frac{1}{4}\ln 7\right)$.

6.3 Applications of the Extreme Value Theorem

Example 1

Suppose two positive numbers sum to 10. What is the largest their product can be?

We can make a chart to try and guess the answer.

First #	Second #	Product
1	9	9
2	8	16
3	7	21
4	6	24
4.5	5.5	24.75
5	5	25

Of course there's no need to continue the chart since we'll just get the same numbers again, though in the opposite order. Also, the numbers don't always have to be integers.

Making a chart is not a rigorous mathematical justification. We'll use what we learned about optimization to show that the largest the product can be is in fact, 25.

Notice that we are not given a function in this problem. This is what makes optimization problems tricky. You have to work out what the function is *before* you start using calculus.

We'll use the notation $f(x)$, so we need to decide just what "x" represents – just like t represented time when we looked at displacement and velocity graphs. Since we're looking for the product of two numbers, we can represent the first number by x.

We might be tempted to say, "Well, let y represent the second number. That way, we can represent the product by $f(x) = x \cdot y$." The only problem with this is that now we have *two* variables – but optimizing with two variables is much more difficult, and doesn't come until Calculus III.

But we're given a bit more information. We know that the sum of the two numbers has to be 10, which we can write as

$$x + y = 10.$$

Now we can solve this for y, which gives up $y = 10 - x$. Plugging back in, we can rewrite our function as

$$f(x) = x(10 - x).$$

Now we have our function. The important observation is that we are asked for the largest product, which means we need a *global maximum.* We just learned a way to find global extrema for continuous functions – and $f(x)$ is a polynomial, so it is continuous.

Another tricky part, though, is the Extreme Value Theorem can be used *only* when the function is defined on a closed interval. So not only do we need to determine a function, we need a reasonable interval to define the function on.

Since the numbers have to be positive, it makes sense to start at 0. What is the upper limit? Well, since the positive numbers have to sum to 10, there is no way one of the numbers can be larger than 10. So a closed interval which makes sense is $[0, 10]$.

What we've done is "translated" the original word problem into an optimization problem: Find the global extrema of the function $f(x) = x(10 - x)$ on the closed interval $[0, 10]$. Let's proceed to apply the three steps to finding these extrema. Keep in mind that we are not asked for a *minimum* value, so we really only have to look for a global maximum in this example.

1. First, find out where $f'(x) = 0$. Note that it is simpler to multiply out $f(x)$ instead of trying to use the Product Rule right away.

$$\begin{aligned} f(x) &= x(10 - x) \\ &= 10x - x^2 \\ f'(x) &= 10 - 2x = 0 \\ x &= 5 \end{aligned}$$

2. Next, evaluate $f(x)$ at these points as well as the endpoints.

$$f(5) = 25, \quad f(0) = 0, \quad f(10) = 0.$$

3. By looking at the values just obtained, we see that there is a global maximum at the point $(5, 25)$, and so 25 is largest possible product.

This seems like a lot of work for just one problem, but it is important to understand *why* we need to take each step. Once we've done it once, we can summarize the process and use it to investigate more examples.

Optimization Strategy

To solve an optimization word problem:

1. Determine what the variable x represents, and *write it down.*
2. Use this to find a function $f(x)$ to optimize. Sometimes it will look like you need two variables (like in the previous example), but you will *always* be able to break it down to just one variable.
3. Find a closed interval which makes sense for the problem.
4. Find the global extrema (whichever you are asked for) using the Extreme Value Theorem.

The main challenge here is that there is no one-size-fits-all method to complete steps (1), (2), and (3). It will be different for each problem. Once we're at step (4), we can use what we learned about the Extreme Value Theorem. This is often the easiest part. So we'll look at some more examples to see how to set up various types of word problems.

Example 2

Suppose you are given a positive number. First, take the square root. Then add 3. Finally, subtract the given number. What is the largest number you can obtain as a result?

Let's look at an example to see what is being asked. If we start with 4, we get $\sqrt{4} = 2$. Then we add 3, giving $2 + 3 = 5$. Finally, subtract 4 (the original number) to get $5 - 4 = 1$.

It turns out we can do a bit better than that. We'll find out how much better by using optimization.

So let's go through the steps one by one.

1. It makes sense to let x represent the positive number you are given.

2. What are we asked to optimize? First we take a number and take the square root, $\sqrt{x}$. Then add $3: \sqrt{x} + 3$. Finally subtract the *given number*, x, giving $\sqrt{x} + 3 - x$. So

$$f(x) = \sqrt{x} + 3 - x.$$

   Note that in this case, there is no need to introduce a second variable.

3. Since x represents a positive number, it makes sense to start at 0. Note that as x gets larger, it becomes greater than $\sqrt{x}$, and is subtracted from $\sqrt{x}$. So $f(x)$ will eventually become negative. When? We know that $f(4) = 1$, which is still positive. But

$$f(9) = \sqrt{9} + 3 - 9 = -3,$$

   so it looks like we can stop at $x = 9$, giving the closed interval $[0, 9]$.

4. So now we have the optimization problem of finding the global maximum of $f(x)$ on the closed interval $[0, 9]$. We have a three-step process to do this.

   (a) When is $f'(x) = 0$?

$$\begin{aligned} f(x) &= \sqrt{x} + 3 - x \\ f'(x) &= \frac{1}{2\sqrt{x}} - 1 = 0 \\ \frac{1}{2\sqrt{x}} &= 1 \\ 2\sqrt{x} &= 1 \\ \sqrt{x} &= \frac{1}{2} \\ x &= \frac{1}{4} \end{aligned}$$

(b) Now evaluate $f(x)$ at this value and the endpoints.

$$f\left(\frac{1}{4}\right) = \sqrt{\frac{1}{4}} + 3 - \frac{1}{4} = \frac{1}{2} + 3 - \frac{1}{4} = \frac{13}{4}, \quad f(0) = 3, \quad f(9) = -3.$$

(c) The largest of these values is $\frac{13}{4}$, so there is a global maximum at $\left(\frac{1}{4}, \frac{13}{4}\right)$. Note that it would be very difficult to guess this value just by making a chart, so we really do need to use calculus here.

Example 3

Now we'll move on to some examples from geometry. Suppose you want to fence in a rectangular area next to a wall, as shown in Figure 6.20. If you have 60 m of fencing, what is the largest area you can enclose?

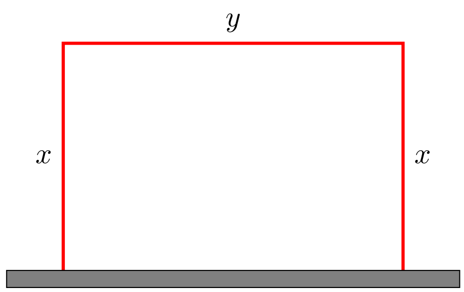


Figure 6.20: A fenced-in area next to a wall.

1. Let x represent the length of one of the sides of the rectangle, and y the length of the other side, as labelled in Figure 6.20. It is important to note that you could have labeled the horizontal side of the rectangle x and the vertical sides y, and you would still get the same answer. The algebra would be different – but you'd still get the same answer. It is often possible to make more than one model for the same word problem.

2. Since we are asked to maximize the area, it makes sense to let $f(x) = x \cdot y$, which is the formula for the area of a rectangle. This gives us two variables again – but since we know we have 60 m of fencing, we know that

$$x + y + x = 60.$$

From this, we get $y = 60 - 2x$, so

$$f(x) = x(60 - 2x) = 60x - 2x^2.$$

Again, it's easier to multiply out so we can use the Power Rule instead of the Product Rule.

3. What closed interval should we choose? Since we have 60 m of fence total, no side can be less that 0 m or greater than 60 m, so $[0, 60]$ would be one choice. But if you see that *two* of the sides of the rectangle have length x, then x cannot be greater than 30 m. So you can also use $[0, 30]$. It's usually easier to work with smaller numbers, so let's use $[0, 30]$.

4. Now we've translated the word problem into the following optimization problem: Find the global maximum of the function $f(x) = 60x - 2x^2$ on the closed interval $[0, 30]$. So we now follow the three steps for solving an optimization problem.

   (a) Observe that $f'(x)$ always exists, since $f(x)$ is a polynomial. We need to find where $f'(x) = 0$.

$$\begin{aligned} f(x) &= 60x - 2x^2 \\ f'(x) &= 60 - 4x = 0 \\ 4x &= 60 \\ x &= 15 \end{aligned}$$

   (b) Now evaluate at this point and the endpoints.

$$f(0) = 0, \quad f(15) = 450, \quad f(30) = 0.$$

   (c) The largest value is 450, so there is a global maximum at $(15, 450)$, and thus the largest possible area is 450 m^2.

Let's see what happens if we labeled the sides of the rectangle differently, as shown in Figure 6.21. This time, x represents the horizontal side of the rectangle.

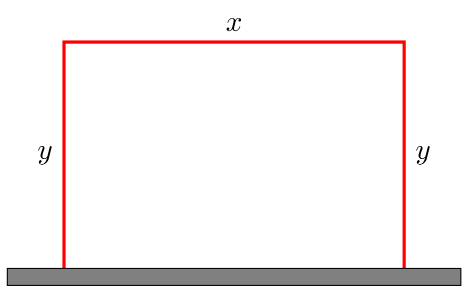


Figure 6.21: A fenced-in area next to a wall.

Note that we still have $f(x) = x \cdot y$, as we are maximizing an area. But this time, we have

$$y + x + y = 60,$$

instead of $x + y + x = 60$. We'll see that the work will be a little different, but the maximum area will still be the same.

Solving for y, we get

$$\begin{aligned} y + x + y &= 60 \\ 2y &= 60 - x \\ y &= 30 - \frac{x}{2} \end{aligned}$$

Substituting back into $f(x)$, we get

$$\begin{aligned} f(x) &= x \cdot y \\ &= x\left(30 - \frac{x}{2}\right) \\ &= 30x - \frac{1}{2}x^2 \end{aligned}$$

Our chosen interval will be different, too. Since there is just one side which is x, we need to use the interval $[0, 60]$. We might have a very thin rectangle with the longer sides being horizontal.

Now we've translated the word problem into the following optimization problem: Find the global maximum of the function $f(x) = 30x - \frac{1}{2}x^2$ on the closed interval $[0, 60]$. Again, we follow the three steps for solving an optimization problem.

1. Observe that $f'(x)$ always exists, since $f(x)$ is a polynomial. We need to find where $f'(x) = 0$.

$$\begin{aligned} f(x) &= 30x - \frac{1}{2}x^2 \\ f'(x) &= 30 - x = 0 \\ x &= 30 \end{aligned}$$

2. Now evaluate at this point and the endpoints.

$$f(0) = 0, \quad f(30) = 450, \quad f(60) = 0.$$

3. The largest value is 450, so there is a global maximum at $(30, 450)$, and thus the largest possible area is 450 m^2. Note that since x represented the length of the horizontal sides of the rectangle, we get a different x-value (this time, 30), but the maximum area is still 450, as before.

Example 4

Suppose you are given a right isosceles triangle whose legs are 2 units long. Inscribe a rectangle in the triangle as shown in Figure 6.22. What is the largest the area of such a rectangle can be?

1. Let x represent the width of the rectangle, as shown in Figure 6.22.
2. Since we are looking to maximize the area of the rectangle, we need a function to represent length $\times$ width. We called x the width – so what is the height? Look at the shaded right isosceles triangle on the right of Figure 6.22. Since the horizontal leg has length x, then the vertical leg has length x as well. So to get the height of the rectangle, we just substract x from 2 to get $2 - x$. Thus, the area of the rectangle is $f(x) = x(2 - x) = 2x - x^2$.
3. Looking at how the rectangle is inscribed in the triangle, we see that an appropriate closed interval for x is $[0, 2]$.

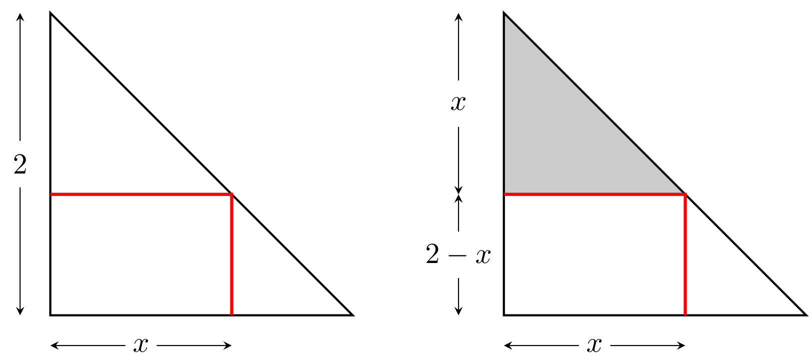


Figure 6.22: Optimizing the area of a rectangle inscribed in a triangle.

4. Now we have the following optimization problem: find the global maximum of $f(x) = 2x - x^2$ on the closed interval $[0, 2]$.

   (a) First, note that $f'(x)$ exists everywhere since $f(x)$ is a polynomial. Now we need to see where $f'(x) = 0$.

   $$\begin{aligned} f(x) &= 2x - x^2 \\ f'(x) &= 2 - 2x = 0 \\ 2x &= 2 \\ x &= 1 \end{aligned}$$

   (b) Now evaluate $f(x)$ at this point and the endpoints.

   $$f(0) = 0, \quad f(1) = 1, \quad f(2) = 0.$$

   (c) Note that 1 is the largest among these values, so there is a global maximum at $(1, 1)$. Thus, the largest possible area is 1 unit^2, in which case the rectangle is actually a square.

Homework:

1. The product of two positive numbers is 16. What is the smallest possible value for their sum?

2. Suppose you start with a positive number. Square it, and then multiply by 2. Then subtract the square root of the given number. What is the smallest result you can obtain by doing this? Note: the answer is a simple fraction.

3. Suppose you have 40 m of fencing and you want to enclose a rectangular region in a corner, as shown in Figure 6.23. What is the largest area you can enclose?

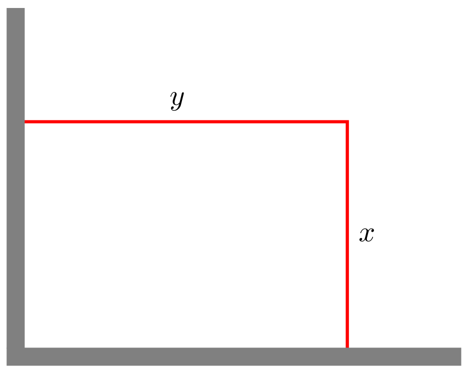


Figure 6.23: A fenced-in area in a corner.

4. Suppose you want to inscribe a rectangle in a right triangle, as shown in Figure 6.24. What is the largest area of such a rectangle? Hint: you will need to look at ratios of corresponding sides of similar triangles for this problem.

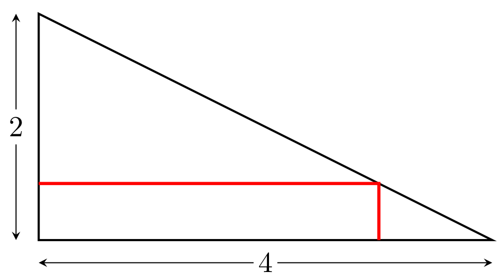


Figure 6.24: Optimizing the area of a rectangle inscribed in a triangle.

Solutions

Problem 1

1. Let x be one of the positive numbers, and let y be the other one.

2. Since we are minimizing the sum, let $f(x) = x+y$. We are given that $xy = 16$, and so $y = \frac{16}{x}$, so that
$$f(x) = x + \frac{16}{x}.$$

3. Since x must be positive but cannot be 0, we choose a small value like 0.1 for the left endpoint. For the right endpoint, we "guesstimate." Now $16 \cdot 1 = 16$ and $16 + 1 = 17$, while $8 \cdot 2 = 16$ and $8 + 2 = 10$. So it looks like the sums getting larger as we from $x = 8$ to 16 and beyond. So we'll choose the interval $[0.1, 16]$.

4. Now optimize $f(x) = x + \frac{16}{x}$ on the closed interval $[0.1, 16]$.

   (a) Then $f(x) = x + 16x^{-1}$, so that $f'(x) = 1 - 16x^{-2}$. Note that $f'(x)$ is always defined (remember, x cannot be 0 since it is not in the interval $[0.1, 16]$).
$$\begin{aligned} f'(x) &= 0 \\ 1 - \frac{16}{x^2} &= 0 \\ 1 &= \frac{16}{x^2} \\ x^2 &= 16 \\ x &= -4, +4 \\ x &= 4 && -4 \text{ is not in the domain} \end{aligned}$$

   (b) Now evaluate at this point and the endpoints.
$$f(0.1) = 160.1, \quad f(4) = 8, \quad f(16) = 17.$$

   (c) The smallest of these values is 8, so 8 is the minimum possible sum.

Problem 2

1. Let x represent the positive number.
2. $f(x) = x^2 \cdot 2 - \sqrt{x} = 2x^2 - x^{1/2}$.
3. $f(0)$ is defined, so we take 0 to be the left endpoint. Let's try a few more values that are easy to calculate:
$$f(4) = 2 \cdot 16 - 2 = 30, \quad f(9) = 2 \cdot 81 - 3 = 159.$$
We'll only get bigger after $x = 9$, so we'll choose our closed interval to be $[0, 9]$.
4. Now optimize $f(x) = 2x^2 - x^{1/2}$ on the closed interval $[0, 9]$. Note that
$$f'(x) = 4x - \frac{1}{2}x^{-1/2} = 4x - \frac{1}{2\sqrt{x}}.$$
   (a) $f'(x)$ is undefined at $x = 0$, but this is an endpoint, so it's already taken care of. Now solve $f'(x) = 0$.
$$\begin{aligned} f'(x) &= 0 \\ 4x - \frac{1}{2\sqrt{x}} &= 0 \\ 4x &= \frac{1}{2\sqrt{x}} \\ 4x \cdot 2\sqrt{x} &= 1 \\ x^{3/2} &= \frac{1}{8} \\ \left(x^{3/2}\right)^{2/3} &= \left(\frac{1}{8}\right)^{2/3} \\ x &= \frac{1}{4} \end{aligned}$$
   (b) Evaluate here and at the endpoints.
$$f(0) = 0, \quad f\left(\frac{1}{4}\right) = -\frac{3}{8}, \quad f(9) = 159.$$
   (c) We conclude that $-\frac{3}{8}$ is the smallest value for $f(x)$, obtained at $x = \frac{1}{4}$.

Problem 3

1. Let x represent the length of one of the sides of the rectangle, and y the length of the other side.

2. Since we are asked to maximize the area, it makes sense to let $f(x) = x \cdot y$. This gives us two variables again – but since we know we have 40 m of fencing, we know that

$$x + y = 40.$$

From this, we get $y = 40 - x$, so

$$f(x) = x(40 - x) = 40x - x^2.$$

3. Since we have 40 m of fence total, no side can be less that 0 m or greater than 40 m, so $[0, 40]$ would be a good choice.

4. Now we've translated the word problem into the following optimization problem: Find the global maximum of the function $f(x) = 40x - x^2$ on the closed interval $[0, 40]$.

   (a) Observe that $f'(x)$ always exists, since $f(x)$ is a polynomial. We need to find where $f'(x) = 0$.

$$\begin{aligned} f(x) &= 40x - x^2 \\ f'(x) &= 40 - 2x = 0 \\ 2x &= 40 \\ x &= 20 \end{aligned}$$

   (b) Now evaluate at this point and the endpoints.

$$f(0) = 0, \quad f(20) = 400, \quad f(40) = 0.$$

   (c) The largest value is 400, so there is a global maximum at $(20, 400)$, and thus the largest possible area is 400 m^2.

Problem 4

1. Let x be the width of the rectangle, as shown in Figure 6.25.

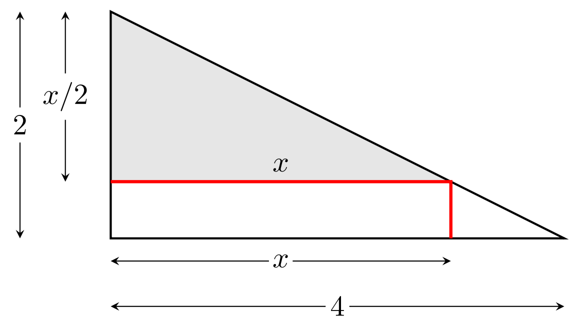


Figure 6.25: Optimizing the area of a rectangle inscribed in a triangle.

2. In the original triangle, the longer leg is twice the shorter leg, and so in the similar gray shaded triangle, the shorter leg must have length $x/2$. This means the height of the rectangle is $2 - \frac{x}{2}$. We are maximizing the area, so we let $f(x)$ be the width times the height, or

$$f(x) = x\left(2 - \frac{x}{2}\right) = 2x - \frac{x^2}{2}.$$

3. Looking at how the rectangle is inscribed in the triangle, we see that an appropriate closed interval for x is $[0, 4]$.

4. Now we have the following optimization problem: find the global maximum of $f(x) = 2x - \frac{x^2}{2}$ on the closed interval $[0, 4]$.

   (a) First, note that $f'(x)$ exists everywhere since $f(x)$ is a polynomial. Now we need to see where $f'(x) = 0$.

$$\begin{aligned} f(x) &= 2x - \frac{x^2}{2} \\ f'(x) &= 2 - x = 0 \\ x &= 2 \end{aligned}$$

   (b) Now evaluate $f(x)$ at this point and the endpoints.

$$f(0) = 0, \quad f(2) = 2, \quad f(4) = 0.$$

   (c) Note that 2 is the largest among these values, so there is a global maximum at $(2, 2)$. Thus, the largest possible area is 2 unit^2.

6.4 Intermediate Value Theorem

We began our discussion of continuity by looking at the behavior of the graph of a function at certain points, such as a jump in the graph. We also needed continuity for the Extreme Value Theorem – necessary for optimization, an important application of calculus. We'll see another use for continuity – showing that equations have solutions in a given interval.

Let's start by looking at part of graph, shown in Figure 8.12.

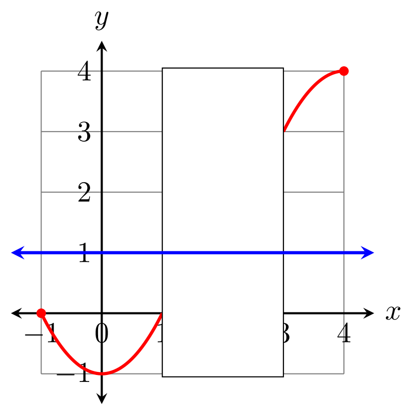


Figure 6.26: Illustrating the Intermediate Value Theorem.

This is the graph of some function on the closed interval $[-1, 4]$, with part of the graph obscured. Will the graph cross the blue line $y = 1$?

Not necessarily. In Figure 6.27, we see a simple way to complete the graph so it does *not* cross the line.

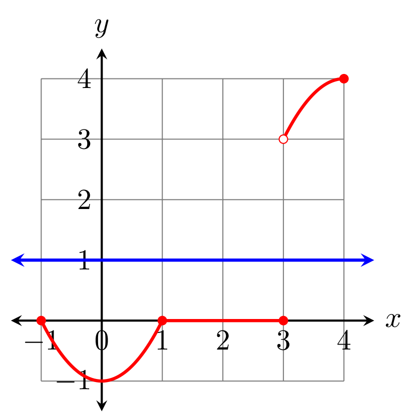


Figure 6.27: Illustrating the Intermediate Value Theorem.

Now let's see where continuity comes into play. Let's ask basically the same question, with a twist: if we assume that the graph in Figure 8.12 is a *continuous* function, does it have to cross the blue line?

The answer to this questions is “Yes.” This is called the Intermediate Value Theorem in calculus. The term “intermediate” is used in the following sense: if we know two different y-values that a continuous function takes on, then it must also take on every y-value between these two values – that is, every intermediate y-value. It is usually stated as follows.

> **Intermediate Value Theorem**
> Suppose $f(x)$ is a continuous function defined on a closed interval $[a, b]$. If $f(a) \neq f(b)$, and if c is between $f(a)$ and $f(b)$, then there is some x_0 in the open interval (a, b) such that $f(x_0) = c$.

This theorem is a way – using calculus terminology – to describe what we observed by looking at graphs. We need this terminology because there is an *infinite* number of ways of drawing continuous graphs between points – and there is no way we can draw an infinite number of graphs and look at them all. But we *can* create a proof using calculus concepts. We won't look at a proof, but it's important to know how to describe using calculus terminology what we visually observe. So let's talk through this theorem using the simplest way of creating a continuous graph – just drawing a straight line through the missing part of the graph.

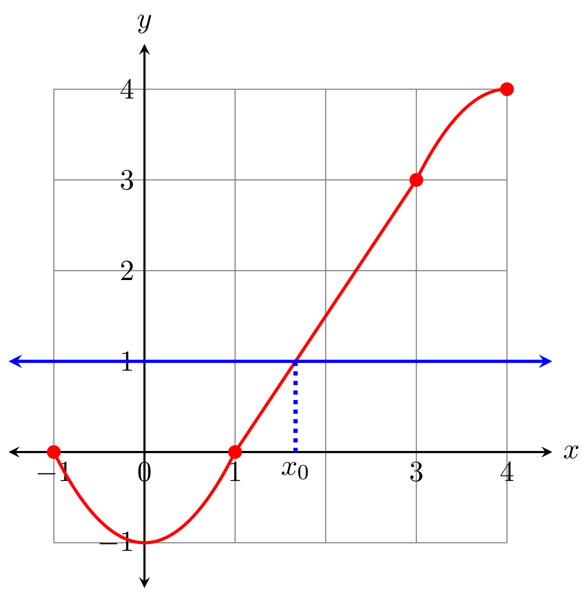


Figure 6.28: Illustrating the Intermediate Value Theorem.

This function is continuous – no jumps, no discontinuities. The closed interval we're looking at is $[a, b] = [-1, 4]$, as we see from the graph. We also observe that $f(-1) = 0$ and $f(4) = 4$, and so $f(-1) \neq f(4)$. Also, $c = 1$ is between 0 and 4. So $x_0 = \dfrac{5}{3}$ is that number in the open interval $(-1, 4)$ such that $f\left(\dfrac{5}{3}\right) = 1$. In Figure 6.29, you can see all these values annotated.

The important point is that the Intermediate Value Theorem doesn't tell you *how* to find this x_0 – it is often quite difficult, requiring a computer or calculator to find. In mathematics, we call this an **existence proof.** It tells you there *is* a thing, but not how to *find* it.

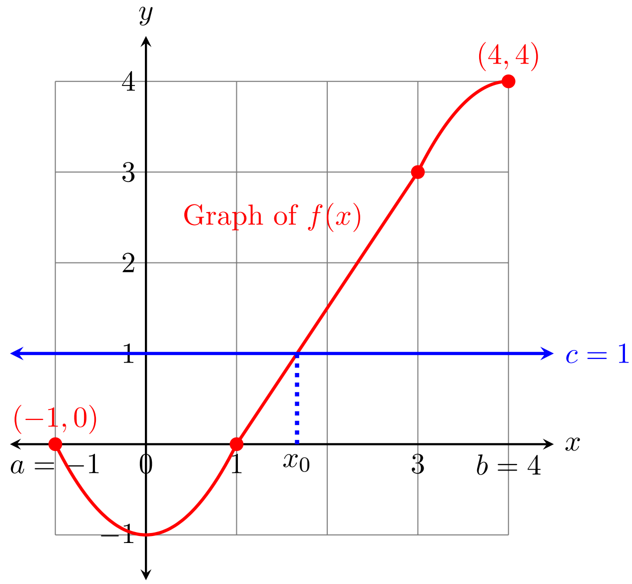


Figure 6.29: Illustrating the Intermediate Value Theorem.

This type of thing happens every day. Think about the stock market. Suppose you have \$1000 to invest. When the stock market opens, you'd like to buy the stock which will have the highest gain at the end of the day. Which one should you buy?

Well, there is one. When the stock market closes, you'd be able to figure it out by looking at the percentage increase or decrease of all the stocks. So when the market opens, there *exists* a stock which will have the highest percentage gain, but at that time, you don't know what it is. You just know that one exists.

This is similar to the Intermediate Value Theorem. It tells you that something *exists,* but it doesn't tell you what it is. Let's look at some examples. We'll use the abbreviation IVT for the Intermediate Value Theorem.

Example 1

Show that the graphs of $y = x$ and $y = \cos(x)$ intersect somewhere in the interval $[0, \pi]$, as shown in Figure 6.30.

How can we use the IVT to show this? We want to show that $x = \cos(x)$ has a solution in the closed interval $[0, \pi]$. The first step is to define a function

$$f(x) = x - \cos(x).$$

Then observe that solving $x = \cos(x)$ is the same as solving $f(x) = 0$. We need a *function* to apply the IVT, so we need to create one. Let's see why this works. We must *also* state that $f(x)$ is continuous as it is a difference of two continuous functions: a basic trigonometric function and a polynomial.

To use the IVT, we need to evaluate $f(x)$ at the endpoints. So

$$f(0) = 0 - \cos(0) = -1, \quad f(\pi) = \pi - \cos(\pi) = \pi + 1 \approx 4.14.$$

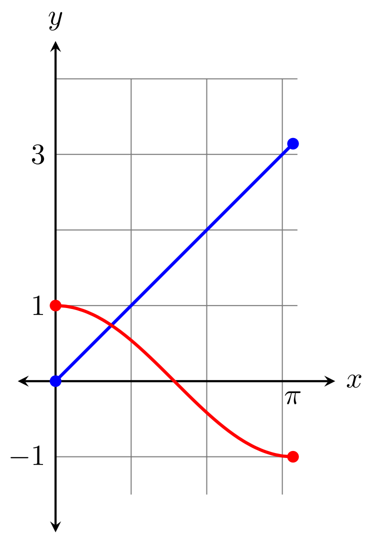


Figure 6.30: Graphs of $y = x$ and $y = \cos(x)$ on the interval $[0, \pi]$.

We remark that $f(0)$ is negative because at $x = 0$, the graph of $y = x$ is *below* the graph of $y = \cos(x)$, and $f(\pi)$ is positive because at $x = \pi$, the graph of $y = x$ is *above* the graph of $y = \cos(x)$.

Now the IVT states that "If $f(a) \neq f(b)$, and if c is between $f(a)$ and $f(b)$, then there is some x_0 in the open interval (a, b) such that $f(x_0) = c$." Here, $a = 0$ and $b = \pi$. Our calculations show that $f(0) \neq f(\pi)$. We'll choose $c = 0$ (since we want to solve $f(x) = 0$), and clearly 0 is between -1 and 4.14. So there must be some x_0 in $[0, \pi]$ with $f(x_0) = 0$. This means that

$$\begin{aligned} f(x_0) &= 0 \\ x_0 - \cos(x_0) &= 0 \\ x_0 &= \cos(x_0) \end{aligned}$$

Thus, the point x_0 is a point where the graphs of $y = x$ and $y = \cos(x)$ intersect. Two important points:

1. The IVT doesn't tell you *where* they intersect. But using a computer, you can approximate the solution to be $x_0 \approx 0.739$. In general, you need a computer to solve equations which combine trigonometric functions and polynomials.

2. There may be *more* than one point of intersection. The IVT tells you that a point *exists,* but doesn't tell you *how many.* To say that such a point exists means there is *at least one* point. There could be more.

It seems obvious that the graphs intersect by looking at them. Keep in mind that the IVT was proved long before the age of computers – you couldn't just type in the equations and have the graphs pop up. Typically, mathematicians use graphs to look for different features of a graph, and use results like the IVT to *prove* that these features do in fact exist.

Here's a summary of the steps to take to show that two curves intersect.

To show that two continuous curves intersect over a given closed interval:

1. Create a function which is the difference of the equations for the curves.
2. Evaluate the function at the endpoints of the closed interval.
3. State that 0 is between these values, so by the IVT, there is some x_0 where the function evaluates to 0.
4. Conclude that the curves intersect at this value x_0.

Example 2

Let's do another example where we use these steps.

Show that the graphs of $y = 4 - x$ and $y = \ln x$ intersect somewhere in the closed interval $[1, 5]$.

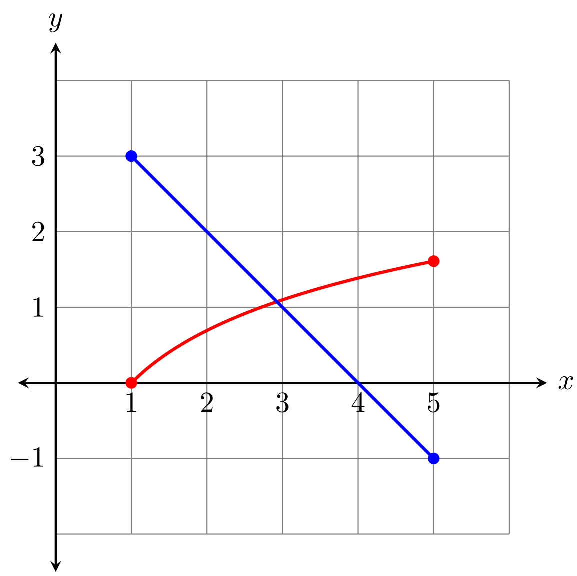


Figure 6.31: Graphs of $y = 4 - x$ and $y = \ln x$ on the interval $[1, 5]$.

1. Define $f(x) = 4 - x - \ln(x)$. When you subtract the functions, it doesn't make sure which order you subtract them in – the logic is the same either way.
2. We calculate:
$$f(x) = 4 - 1 - \ln 1 = 3, \quad f(5) = 4 - 5 - \ln 5 \approx -2.61.$$
3. Since 0 is between 3 and -2.61, by the IVT there is some value of x_0 between 1 and 5 such that $f(x_0) = 0$.
4. Since $f(x)$ was defined as the difference between the equations for the graphs, they intersect at this point x_0. You need a calculator or computer to find that $x_0 \approx 2.93$.

Homework

1. Show that the graphs of the curves $y = e^{2x}$ and $y = 4 - x^2$ intersect in the closed interval $[0, 2]$.
2. Show that the graphs of the curves $y = \dfrac{\ln x}{x}$ and $y = e^x - 5$ intersect in the closed interval $[1, 2]$.
3. Show that the graphs of the curves $y = \sin(2x)$ and $y = 1 - \cos(x)$ intersect three times in the closed interval $[0, 2\pi]$. Hint: Graph these functions on desmos. You should only have to use the IVT once, but you'll see that you can't use the closed interval $[0, 2\pi]$ when you apply the IVT.

Solutions

Problem 1

1. Let $f(x) = e^{2x} - (4 - x^2) = e^{2x} - 4 + x^2$.
2. Evaluate:
$$f(0) = -3, \quad f(2) \approx 54.6.$$
3. Note that $-3 < 0 < 54.6$, so by the IVT, there is some x_0 where $f(x_0) = 0$.
4. This means that the curves intersect when $x = x_0$ (and possibly other points).

Problem 2

1. Let $f(x) = \dfrac{\ln x}{x} - (e^x - 5) = \dfrac{\ln x}{x} - e^x + 5$.
2. Evaluate at the endpoints.
$$f(1) \approx 2.28, \quad f(2) \approx -2.02.$$
3. Note that $-2.02 < 0 < 2.28$, so by the IVT, there is some x_0 where $f(x_0) = 0$.
4. This means that the curves intersect when $x = x_0$ (and possibly other points).

Problem 3

1. Let $f(x) = \sin(2x) - (1 - \cos(x)) = \sin(2x) - 1 + \cos(x)$.
2. By looking at the graphs on desmos, it appears that the curves intersect at $x = 0$ and $x = 2\pi$. It is not difficult to evaluate $f(0) = 0$ and $f(2\pi) = 0$, meaning that the curves intersect at $x = 0$ and $x = 2\pi$. It looks like there is a third intersection somewhere between $\dfrac{\pi}{4}$ and $\dfrac{\pi}{2}$, so we use the closed interval $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$. We now evaluate at the endpoints of this interval.
$$f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{2}\right) - 1 + \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad f\left(\frac{\pi}{2}\right) = \sin(\pi) - 1 + \cos\left(\frac{\pi}{2}\right) = -1.$$
3. Note that $-1 < 0 < \dfrac{1}{\sqrt{2}}$, so by the IVT, there is some x_0 where $f(x_0) = 0$.
4. This means that the curves intersect when $x = x_0$ (and possibly other points).

Chapter 7

Asymptotes and Infinity

7.1 Asymptotes and Limits to Infinity, I

We have described many features of graphs up to the point using limits and calculus. The last features we will describe are horizontal and vertical asymptotes. We'll look at some new notation by examining the graph of $f(x) = \dfrac{1}{x}$.

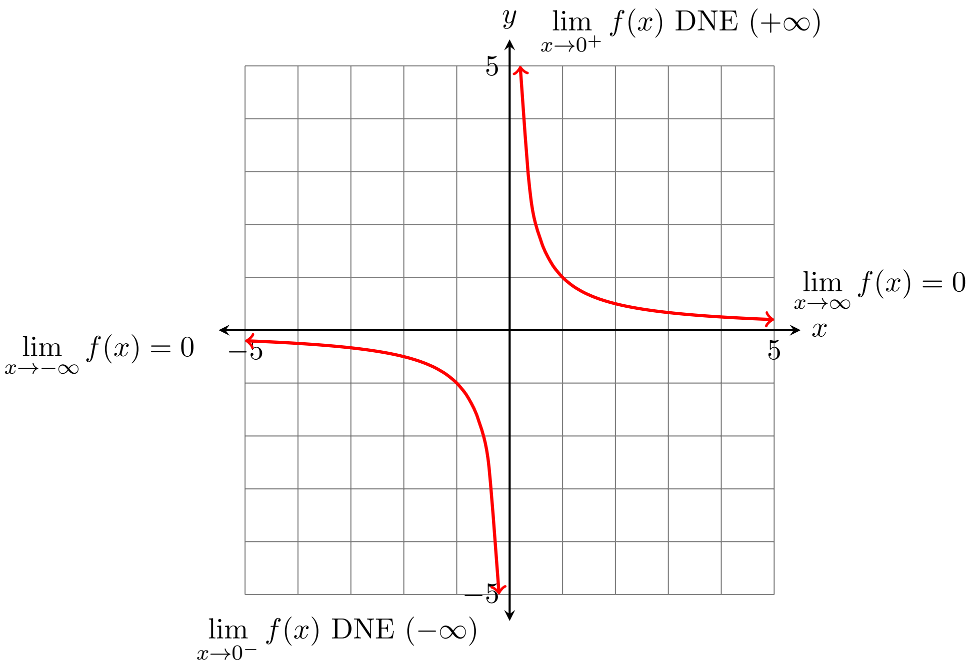


Figure 7.1: Graph of $f(x) = \dfrac{1}{x}$ with asymptotes described.

Let's see what this new notation means.

1. The $+x$ axis. We read this as "the limit as x goes to infinity of $f(x)$ is 0." Graphically, this means that $f(x)$ is getting closer and closer to 0 as x gets further and further along the x axis. We can also see this using a chart of values, like we did when taking $\lim_{h\to 0} \dfrac{\sin(h)}{h}$.

x	$1/x$
1	1
10	0.1
100	0.01
1000	0.001
10,000	0.0001

Table 7.1: Looking at $\lim_{x\to\infty} f(x) = 0$.

We will look at rules for evaluating such limits later, but the chart is a numerical confirmation that this limit is 0. This means that $y = 0$ is a horizontal asymptote as we look to the right.

2. The $+y$ axis. We read this as "The limit as x goes to 0 from the right does not exist, but it approaches positive infinity," meaning the values of the function keep getting larger and larger without bound.

   Again, we look at a numerical chart to see what's happening. We can see that the values of

x	$1/x$
1	1
0.1	10
0.01	100
0.001	1000
0.0001	10,000

Table 7.2: Looking at $\lim_{x \to 0^+} f(x)$ DNE $(+\infty)$.

   $f(x)$ will keep getting larger the smaller that x gets. So there is *no* limiting value – the limit does not exist. The notation "$(+\infty)$" indicates that the graph moves *up* along the positive y axis, but never actually touches it.

3. The $-x$ axis. We read this as "The limit as x approaches negative infinity of $f(x)$ is 0." There is no need to make a chart here; it will look very similar to the chart for the $+x$ axis. This means $y = 0$ is also a horizontal asymptote as we look to the left. It is important to observe that here, the curve approaches the asymptote from *below,* but at the $+x$ axis, the curve approaches the asymptote from *above.* The notatation itself does *not* tell you if the curve approaches from above or below, so you need to do additional work to decide which.

4. The $-y$ axis. We read this as "The limit as x approaches 0 from the left does not exist, but it approaches negative infinity." Again, a chart would look very similar to that for the $+y$ axis. As x moves closer and closer to 0 from the left, the graph approaches the y axis, and keeps going *down* along the y axis, with no lower bound. Thus, there is no lower limit – this is what "DNE $(-\infty)$" means.

The types of functions we'll be looking at are **rational functions,** which are ratios of polynomials; $f(x) = \dfrac{1}{x}$, for example. That is, the numerator and denominator of the function are both polynomials. We'll state how to find asymptotes of such functions, and then look at several examples. Remember that the **degree** of a polynomial is the highest power of x occurring in a polynomial, and that the degree of a constant polynomial (like $f(x) = 5$) is 0, since $5 = 5 \cdot x^0$. Also, the **leading coefficient** of a polynomial is the coefficient of the term with highest degree. So the leading coefficient of $3x^4 - 2x^x + 5x$ is 3.

Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function. Let N be the degree of $p(x)$, and D be the degree of $q(x)$. To find the horizontal asymptotes:

1. If $N > D$, there are no horizontal asymptotes.
2. If $N = D$, then there is a horizontal asymptote, in both directions, at $y = c$, where c is the ratio of the leading coefficients of the numerator and denominator.
3. If $N < D$, there is a horizontal asymptote, in both directions, at $y = 0$.

To find the vertical asymptotes, first cancel out common factors of the numerator and denominator, if any. Then there are vertical asymptotes where the denominator is equal to 0.

We'll look at several examples which illustrate all of these possibilities.

Example 1

The graph of $f(x) = \dfrac{1}{x^2}$ is shown in Figure 7.2.

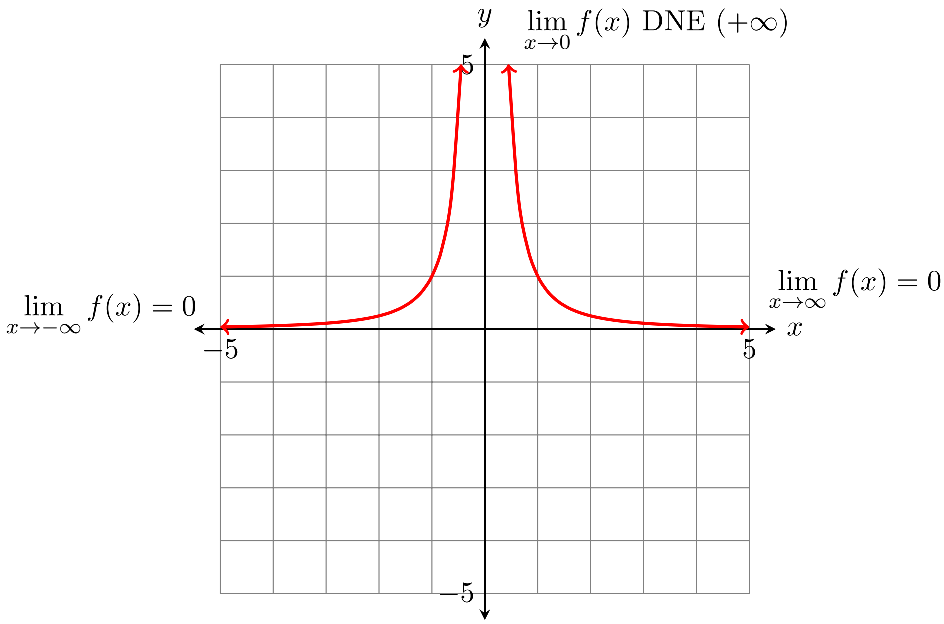


Figure 7.2: Graph of $f(x) = \dfrac{1}{x^2}$ with asymptotes described.

Let's apply our method here. For the numerator, $N = 0$ since a constant has degree 0, and for the denominator $D = 2$. Since $N < D$, $y = 0$ is a horizontal asymptote.

Now let's look at vertical asymptotes. There are no factors to cancel, so we look at when $x^2 = 0$, which is when $x = 0$. Thus, there is a vertical asymptote at $x = 0$. Because $f(x)$ must always be positive, then

$$\lim_{x\to 0^-} f(x) \text{ DNE } (+\infty), \quad \lim_{x\to 0^+} f(x) \text{ DNE } (+\infty).$$

Since we have DNE $(+\infty)$ from the left and the right, we can just write

$$\lim_{x\to 0} f(x) \text{ DNE } (+\infty).$$

This is similar to how we used the limit notation when looked at one-sided limits earlier. It is important to note that this notation describes the asymptotic behavior at $x = 0$, but it does *not* mean that the limits exist. Remember, DNE means "does not exist."

Example 2

The graph of $f(x) = \dfrac{4x}{x^2+1}$ is shown in Figure 7.3.

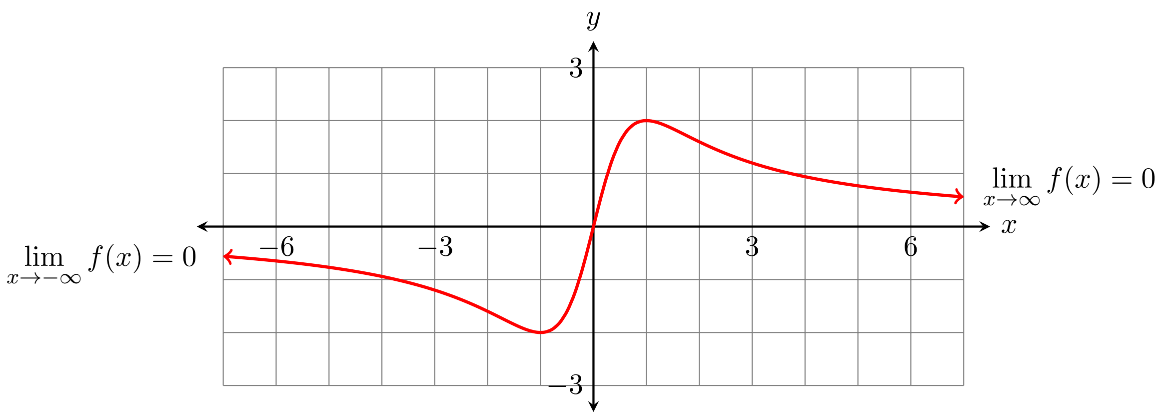


Figure 7.3: Graph of $f(x) = \dfrac{4x}{x^2+1}$ with asymptotes described.

Let's apply our method here. For the numerator, $N = 1$, and for the denominator $D = 2$. Since $N < D$, $y = 0$ is a horizontal asymptote. Let's take a moment to see why.

As $x \to \infty$, the denominator – having a higher degree – blows up faster than the numerator. This forces the fraction to 0. A few values support this, as seen in the following chart. $f(1) = 2$, but other values are approximate.

x	$f(x)$
1	2
10	0.4
100	0.04
1000	0.004
10,000	0.0004

Table 7.3: Looking at $\lim_{x\to\infty} f(x) = 0$.

How do we know the graph approaches $y = 0$ from above here? Note that for large, positive x,

$$f(x) = \frac{+}{+} = + > 0,$$

and so we approach from above.

A chart as $x \to -\infty$ looks similar. The graph approaches $y = 0$ from below since for negative x,

$$f(x) = \frac{-}{+} = - < 0,$$

which means we approach from below. These calculations are necessary if you do not have a graph of the function.

Since the denominator can never be 0 ($x^2 + 1$ is always positive), there are no vertical asymptotes.

Example 3

The graph of $f(x) = \dfrac{3x^2 + x}{4x^2 - 4}$ is shown in Figure 7.4. There is a lot going on here, so let's go one step at a time.

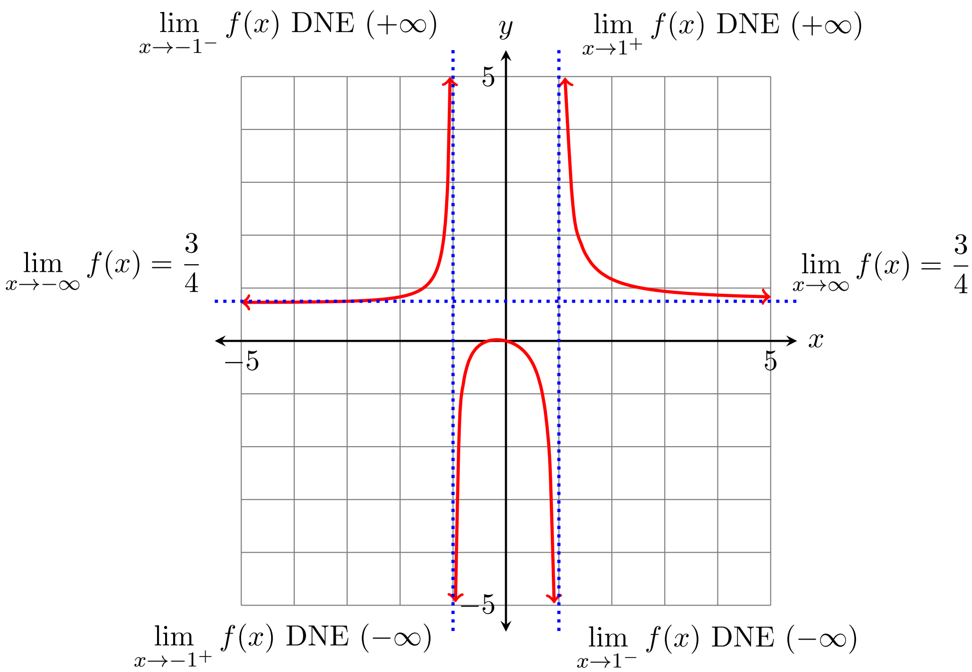


Figure 7.4: Graph of $f(x) = \dfrac{3x^2 + x}{4x^2 - 4}$ with asymptotes described.

The degree of the numerator is $N = 2$, and the degree of the denominator is $D = 2$. Since $N = D$, we have a horizontal asymptote at the ratio of the leading coefficients, or $y = \dfrac{3}{4}$. Let's see why this makes sense.

The terms with the highest power of x take over as $x \to \infty$. In other words we can say that for large x,

$$f(x) = \frac{3x^2 + x}{4x^2 - 4} \approx \frac{3x^2}{4x^2} = \frac{3}{4}.$$

A brief table of function values (see Table 7.5) supports this. You can see how the function values approach $\dfrac{3}{4} = 0.75$. Also observe that for values of x far to the left or right, $f(x) = \dfrac{+}{+} = +$, so the graph approaches the horizontal asymptote from above in both directions.

x	$f(x)$
1000	0.750251
10,000	0.750025
100,000	0.750003

Table 7.4: Looking at $\lim_{x\to\infty} f(x) = \frac{3}{4}$.

Now let's find the vertical asymptotes. First, note that we cannot cancel any factors, since

$$f(x) = \frac{3x^2 + x}{4x^2 - 4} = \frac{x(3x+1)}{4(x+1)(x-1)}.$$

So now see where the denominator is 0, which is when $x = -1$ or $x = 1$. Thus, the lines $x = -1$ and $x = 1$ are vertical asymptotes.

From the graph, it is easy to see the limits as x approaches 1 from the left and right. But suppose you didn't have a graph? We make a brief table of approximate values with numbers close to 1 on either side. These values make the trend clear. Note that the graph approaches the asymptote in

x	$f(x)$
0.99	-49.3756
0.999	-499.375
1.01	50.6256
1.001	500.625

Table 7.5: Looking at $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

different directions at $x = 1$, and so we write

$$\lim_{x\to 1} f(x) \text{ DNE.}$$

Contrast this to Example 1, where we were able to write

$$\lim_{x\to 0} f(x) \text{ DNE } (+\infty).$$

We observe similar behavior at $x = -1$, and so skip the details.

Example 4

The graph of $f(x) = \dfrac{2x^2 - 5}{2x - 3}$ is shown in Figure 7.6.

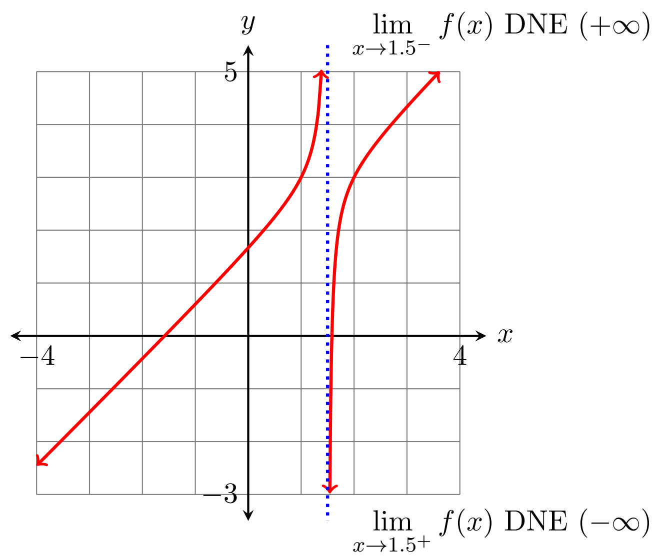


Figure 7.5: Graph of $f(x) = \dfrac{2x^2 - 5}{2x - 3}$ with asymptotes described.

Let's start with the horizontal asymptotes. The degree of the numerator is $N = 2$ and the degree of the denominator is $D = 1$. Since $N > D$, there are no horizontal asymptotes. This makes sense, since as x moves further out in either direction, we approximate

$$f(x) = \frac{2x^2 - 5}{2x - 3} \approx \frac{2x^2}{2x} = x.$$

Thus, there is no limiting value for $f(x)$ as $x \to \infty$ or $x \to -\infty$.

Now let's look at vertical asymptotes. The only candidate is $x = 3/2$, since this is where the denominator is 0. We do need to check if any factors cancel, though. Now if the $2x - 3$ *did* cancel, we'd need a factor of $2x - 3$ in the numerator. This means that if we plug $x = 3/2$ into the numerator, we'd *also* have to get 0. But

$$2\left(\frac{3}{2}\right)^2 - 5 = -\frac{1}{2} \neq 0,$$

so there is no cancellation. As with Example 3, we can make a brief chart of values to see the behavior of the graph as it approaches the asymptote $x = \dfrac{3}{2}$. We'll skip that here.

Example 5

The graph of $f(x) = \dfrac{x^2-1}{x-1}$ is shown in Figure 7.6.

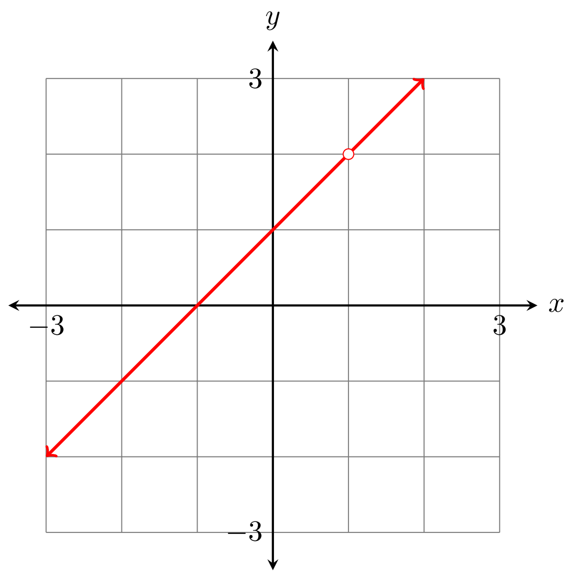


Figure 7.6: Graph of $f(x) = \dfrac{x^2-1}{x-1}$.

Let's first look for horizontal asymptotes. The degree of the numerator is $N = 2$ and the degree of the denominator is $D = 1$. Since $N > D$, there are no horizontal asymptotes.

At first glance, it looks like there is a vertical asymptote at $x = 1$. But in this case, the numerator factors, and so

$$f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1.$$

So it looks like the graph of $f(x)$ is just a straight line, $y = x + 1$. This is almost correct – but we can't skip over the fact that the original function is not defined at $x = 1$, since we'd get $\frac{0}{0}$. So we need to put an open circle at $(1, 2)$. This is because we can use the formula $x + 1$ *only* when $x \neq 1$, since $f(x)$ is undefined at $x = 1$.

Homework

1. For each of the following functions, first graph it is desmos, then find all asymptotes. Describe the behavior at the asymptotes using limit notation, as in the examples in the notes.

   (a) $f(x) = \dfrac{x}{x^2 - 4}$

   (b) $f(x) = \dfrac{x^3 - 1}{x^3 + 1}$

   (c) $f(x) = \dfrac{x^2 - x - 2}{x^2 - 1}$

2. For the function in 1(b), show how you would describe the behavior at the vertical asymptote if you did *not* have a graph.

3. For the function in 1(c), show how you would know if the graph approached the horizontal asymptote from above or below as $x \to -\infty$ and $x \to \infty$ if you did *not* have a graph.

Solutions

1. (a) Since $N = 1$ and $D = 2$ and $N < D$, there is a horizontal asymptote at $y = 0$. We write

$$\lim_{x\to-\infty} f(x) = 0, \quad \lim_{x\to\infty} f(x) = 0.$$

No factors cancel out. Since the denominator is 0 when $x = -2, 2$, we have vertical asymptotes at $x = -2$ and $x = 2$. Writing using limits, we have

$$\lim_{x\to-2^-} f(x) \text{ DNE } (-\infty), \quad \lim_{x\to-2^+} f(x) \text{ DNE } (+\infty),$$

and

$$\lim_{x\to2^-} f(x) \text{ DNE } (-\infty), \quad \lim_{x\to2^+} f(x) \text{ DNE } (+\infty).$$

(b) Here, $N = 3$ and $D = 3$, and since $N = D$, there is a horizontal asymptote at $y = \frac{1}{1} = 1$. We have

$$\lim_{x\to-\infty} f(x) = 1, \quad \lim_{x\to\infty} f(x) = 1.$$

The denominator is 0 only when $x^3 = -1$, or $x = -1$. Plugging -1 into the numerator, we get $(-1)^3 - 1 = -2 \neq 0$, so we know nothing cancels. In this case, we have

$$\lim_{x\to-1^-} f(x) \text{ DNE } (+\infty), \quad \lim_{x\to-1^+} f(x) \text{ DNE } (-\infty).$$

(c) Here, $N = 2$ and $D = 2$, and since $N = D$, there is a horizontal asymptote at $y = \frac{1}{1} = 1$. We have

$$\lim_{x\to-\infty} f(x) = 1, \quad \lim_{x\to\infty} f(x) = 1.$$

First, we factor, getting

$$f(x) = \frac{(x-2)(x+1)}{(x-1)(x+1)} = \frac{x-2}{x-1}.$$

Once we cancel, we see the denominator is 0 only when $x = 1$, which is where the vertical asymptote is. We have

$$\lim_{x\to1^-} f(x) \text{ DNE } (+\infty), \quad \lim_{x\to1^+} f(x) \text{ DNE } (-\infty).$$

2. The vertical asymptote is at $x = -1$. So we make a brief chart getting close to -1 from the left and from the right.

x	$f(x)$
-1.01	67.0044
-1.001	667.000
-0.99	-66.3378
-0.999	-666.334

Table 7.6: Looking at $\lim_{x \to -1^-} f(x)$ and $\lim_{x \to -1^+} f(x)$.

So from the left, we are tending toward $(+\infty)$, and from the right, we are tending toward $(-\infty)$.

3. We make a brief chart with values of x to the left and to the right.

x	$f(x)$
-100	1.0099
-1000	1.001
100	0.989899
1000	0.998999

Table 7.7: Looking at $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$.

So as $x \to -\infty$, we see the values are a little bigger than 1, and so the graph approaches the asymptote from above. As $x \to \infty$, we see the values are a little smaller than 1, and so the graph approaches the asymptote from below.

7.2 Asymptotes and Limits at Infinity, II

For the initial discussion, you will need to visit desmos.com.

We start our discussion by looking at the *growth of functions.* For example, given $f(x) = e^x$ and $g(x) = x^2$, we know that $\lim_{x\to\infty} f(x)$ DNE $(+\infty)$ and $\lim_{x\to\infty} g(x)$ DNE $(+\infty)$. In other words, both functions "blow up" as $x \to \infty$. Is there a way to tell which function blows up faster?

Note: the different lines/functions in the desmos graph are labelled ◯1–◯10. You can select/deselect by clicking on the circles to the left. When you open the graph, you should see ◯2 and ◯3 selected.

The graph of $f(x) = e^x$ is shown. As you move the slider in ◯1, you'll graphs of $g(x) = x$, $g(x) = x^2$, and so on, up to $g(x) = x^{10}$. Once you hit $g(x) = x^3$, it looks like the polynomial is "winning." If you click on the wrench in the upper right and deselect "Lock Viewport," you'll be able to zoom out. If you do, you'll notice it's impossible to tell which function is growing faster.

How can we use limits and calculus to see which function grows faster? One way is to look at the quotient of $f(x)$ and $g(x)$. In other words, we consider $h(x) = \dfrac{x^2}{e^x}$. Make sure that *only* ◯3, ◯4, and ◯5 are selected. Click on the house icon in the upper right to reset the screen.

Now make sure you can tell which graphs are $f(x) = e^x$, $g(x) = x^2$, and $h(x) = \dfrac{x^2}{e^x}$. From looking at the graph, we might guess that

$$\lim_{x\to\infty} h(x) = 0.$$

How can we show this? When we looked at rational functions, we had a definite procedure to follow. When other functions are involved in the quotient, we use a method called **L'Hôpital's Rule**. This method is used for quotients of the form "$\dfrac{\pm\infty}{\pm\infty}$," which means the limits in the numerator and denominator do not exist, but tend towards positive or negative infinity – that is, the limits of the numerator and denominator are either DNE $(+\infty)$ or DNE $(-\infty)$. Here is the result.

> Suppose $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is of the form $\dfrac{\pm\infty}{\pm\infty}$. Then
>
> $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$
>
> Here, a can be a number or $\pm\infty$.

Let's see how we apply this here. We observed that $\lim_{x\to\infty} x^2$ DNE $(+\infty)$ and $\lim_{x\to\infty} e^x$ DNE $(+\infty)$. So the form is right for using L'Hôpital's Rule. Thus,

$$\lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{2x}{e^x},$$

where the notation "$\overset{\text{LR}}{=}$" means that the expressions are equal due to L'Hôpital's Rule.

Now you'll notice that $\lim_{x\to\infty} \frac{2x}{e^x}$ is still of the form $\frac{\pm\infty}{\pm\infty}$. So that means we'll have to use L'Hôpital's Rule a second time. This is not unusual when using L'Hôpital's Rule. Therefore,

$$\lim_{x\to\infty} \frac{2x}{e^x} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{2}{e^x}.$$

But now we observe that the numerator is *always* 2, but the denominator tends to $+\infty$. Therefore,

$$\lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

So this means that $h(x) = \frac{x^2}{e^x}$ does have a horizontal asymptote at $y = 0$. But *only* in the positive direction. From the graph, it appears that $\lim_{x\to-\infty}$ DNE $(+\infty)$. This is because as $x \to -\infty$, the numerator x^2 keeps getting larger and larger, while the denominator e^x goes to 0. Recall that with rational functions, the function always approached the horizontal asymptote from both directions. As we see here, that is not necessarily the case if the quotient is not a rational function.

Essentially, because we take derivatives on the numerator and denominator, L'Hôpital's Rule is saying that to look at the ratio of two expressions tending to $\pm\infty$, we need to compare the *rates* at which these expressions tend to $\pm\infty$.

Now that we have L'Hôpital's Rule, it is important *not* to confuse it with the quotient rule. Remember, to take the derivative of a quotient, you *cannot* just take the derivative of the numerator and the derivative of the denominator. But that's *exactly* what you do when using L'Hôpital's Rule.

> You can use L'Hôpital's Rule to evaluate limits of the form $\frac{\pm\infty}{\pm\infty}$.
> You *cannot* use L'Hôpital's Rule when taking derivatives.

Example 1

We can ask a similar question about $f(x) = \ln x$ and $g(x) = \sqrt{x}$. Make sure only ◯6 and ◯7 are selected. It looks like $g(x)$ is above $f(x)$ when the home screen icon is clicked. But what about

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}?$$

The logarithm is below the square root so far, but what happens later? If you select ◯8 and zoom out, it looks like this ratio could be 0, or perhaps a small number. We can use L'Hôpital's Rule to see just what happens, since for both graphs, $\lim_{x\to\infty} f(x)$ DNE $(+\infty)$ and $\lim_{x\to\infty} g(x)$ DNE $(+\infty)$.

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})}.$$

This is starting to look complicated, but when using L'Hôpital's Rule (perhaps more than once), it is important to simplify first. Thus,

$$\frac{1/x}{1/(2\sqrt{x})} = \frac{1}{x} \cdot \frac{2\sqrt{x}}{1} = \frac{2}{\sqrt{x}}.$$

Then we see that $\lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$, since the denominator goes to infinity while the numerator stays at 2. Thus, we conclude that

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0.$$

Let's see geometrically why this makes sense. Select ◯3, ◯4, ◯6, and ◯7 only. Notice that e^x and $\ln x$ are inverse functions, as are x^2 (for $x \geq 0$) and $\sqrt{x}$. So if e^x grows *faster* that x^2 as $x \to \infty$, then when you reflect about the line $y = x$ (this is how you transform a graph to get the graph of an inverse function), then $\ln x$ grows *slower* that $\sqrt{x}$.

Example 2

There is a second way we can apply L'Hôpital's Rule. In the previous examples, we did so when the limit was of the form $\dfrac{\pm\infty}{\pm\infty}$. We can also use L'Hôpital's Rule when the limit is of the form $\dfrac{0}{0}$. Said using limits, if we want to find $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}$, and if both $\lim\limits_{x\to 0} f(x) = 0$ and $\lim\limits_{x\to 0} g(x) = 0$, then L'Hôpital's Rule can be used.

Let's look at

$$\lim_{h\to 0}\frac{\cos(h)-1}{h}.$$

We used this limit when we used the definition to find the derivative of $\sin(x)$. Let's see how to find the limit using L'Hôpital's Rule.

Since $\cos(0) = 1$, then $\lim\limits_{h\to 0}(\cos(h) - 1) = 0$, and of course $\lim\limits_{h\to 0} h = 0$. This means the limit is of the form $\dfrac{0}{0}$. So

$$\lim_{h\to 0}\frac{\cos(h)-1}{h} \overset{\text{LR}}{=} \lim_{h\to 0}\frac{-\sin(h)}{1} = \lim_{h\to 0}(-\sin(h)).$$

There is no longer a fraction involved, and $\sin(0) = 0$, so this limit is 0; that is

$$\lim_{h\to 0}\frac{\cos(h)-1}{h} = 0.$$

We did say how this could be shown numerically and graphically before, but we can verify it using L'Hôpital's Rule.

Example 3

Now consider a similar limit,

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x^2}.$$

Like the previous example, we observe that this limit is of the form $\frac{0}{0}$. Therefore, we may apply L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x^2} \stackrel{\text{LR}}{=} \lim_{x\to 0} \frac{-\sin(x)}{2x}.$$

No need to simplify here, but we need to check what form the limit is. Again, we see that it is of the form $\frac{0}{0}$, so we can apply L'Hôpital's Rule again.

$$\lim_{x\to 0} \frac{-\sin(x)}{2x} \stackrel{\text{LR}}{=} \lim_{x\to 0} \frac{-\cos(x)}{2}.$$

This time around, nothing is 0, so we can just plug in $x = 0$ here, so

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x^2} = -\frac{1}{2}.$$

What does it mean that we've calculated this limit? This means that very near 0, $\frac{\cos(x) - 1}{x^2}$ is approximately $-\frac{1}{2}$, which we write as

$$\frac{\cos(x) - 1}{x^2} \approx -\frac{1}{2}.$$

Now let's "solve" this for $\cos(x)$ as follows.

$$\begin{aligned} \frac{\cos(x) - 1}{x^2} &\approx -\frac{1}{2} \\ \cos(x) - 1 &\approx -\frac{1}{2} \cdot x^2 \\ \cos(x) &\approx 1 - \frac{x^2}{2}. \end{aligned}$$

Now go back to desmos, and make sure only ◯9 and ◯10 are selected. If you go back to the home screen, you'll see that these functions look very different. But as you zoom in around $x = 0$, you should notice some interesting behavior. The graphs seem to get closer and closer together, and if you zoom in far enough, you can't tell the difference between the two – they look like they're on top of each other. They *only* intersect at $x = 0$, but there's no way to tell this using your computer. It doesn't have good enough resolution for that.

The point is that we discovered that $y = 1 - \frac{x^2}{2}$ is a very good approximation to $y = \cos(x)$ by using L'Hôpital's Rule. Finding approximations is such an important application of calculus that one-third of Calculus II is devoted to this topic. We can only scratch the surface here. But as these examples show, using L'Hôpital's Rule is yet another way to study the behavior of graphs.

Summary of L'Hôpital's Rule:

> Suppose $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of the form $\frac{\pm\infty}{\pm\infty}$ or of the form $\frac{0}{0}$. Then L'Hôpital's Rule tells us that
>
> $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$
>
> Here, a can be a number or $\pm\infty$. L'Hôpital's Rule can *never* be used to evaluate a derivative!

In-Class Practice/Homework

For each of the limits below, decide whether you would be able to apply L'Hôpital's Rule to evaluate. Do *not* evaluate the limits now. For homework, evaluate all the limits where L'Hôpital's Rule may be applied.

1. $\lim_{x\to\infty} \frac{e^x}{x^3+1}$
2. $\lim_{x\to-\infty} \frac{e^x}{x^3+1}$
3. $\lim_{x\to\infty} \frac{x+2}{x^2-4}$
4. $\lim_{x\to 2} \frac{x+2}{x^2-4}$
5. $\lim_{x\to\infty} \frac{\ln x}{e^{-x}}$
6. $\lim_{x\to 0} \frac{\sin(x)}{x^2+x}$
7. $\lim_{x\to-\infty} \frac{\sin(x)}{x^2+x}$
8. $\lim_{x\to\pi} \frac{\sin(x)}{1+\cos(x)}$
9. $\lim_{x\to 0} \frac{\sin(x)}{1+\cos(x)}$
10. $\lim_{x\to 1} \frac{\sqrt{x}-1}{x^2-1}$
11. $\lim_{x\to\infty} \frac{\sqrt{x}-1}{x^2-1}$
12. $\lim_{x\to-\infty} \frac{\sqrt{x}-1}{x^2-1}$
13. $\lim_{x\to\infty} \frac{\ln(\ln x))}{x^2+1}$
14. $\lim_{x\to\infty} \frac{e^{-x}}{x^2}$
15. $\lim_{x\to-\infty} \frac{e^{-x}}{x^2}$

Solutions

1. The limit is of the form $\dfrac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{e^x}{x^3+1}\overset{\text{LR}}{=}\lim_{x\to\infty}\frac{e^x}{3x^2}\overset{\text{LR}}{=}\lim_{x\to\infty}\frac{e^x}{6x}\overset{\text{LR}}{=}\lim_{x\to\infty}\frac{e^x}{6}\text{ DNE }(+\infty).$$

2. The limit is of the form $\dfrac{0}{-\infty}$, so L'Hôpital's Rule does not apply.

3. The limit is of the form $\dfrac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{x+2}{x^2-4}\overset{\text{LR}}{=}\lim_{x\to\infty}\frac{1}{2x}=0.$$

4. The limit is of the form $\dfrac{4}{0}$, so L'Hôpital's Rule does not apply.

5. The limit is of the form $\dfrac{\infty}{0}$, so L'Hôpital's Rule does not apply.

6. The limit is of the form $\dfrac{0}{0}$.

$$\lim_{x\to0}\frac{\sin(x)}{x^2+x}\overset{\text{LR}}{=}\lim_{x\to0}\frac{\cos(x)}{2x+1}=1.$$

7. The function $y=\sin(x)$ oscillates continuously between -1 and 1, and so has no limit as $x\to\infty$. So L'Hôpital's Rule cannot be used.

8. The limit is of the form $\dfrac{0}{0}$.

$$\lim_{x\to\pi}\frac{\sin(x)}{1+\cos(x)}\overset{\text{LR}}{=}\lim_{x\to\pi}\frac{\cos(x)}{-\sin(x)}\text{ DNE }.$$

9. The limit is just $\dfrac{0}{2}=0$. You can just plug in here, and since the denominator tends to 2, L'Hôpital's Rule cannot be applied.

10. The limit is of the form $\dfrac{0}{0}$.

$$\lim_{x\to1}\frac{\sqrt{x}-1}{x^2-1}\overset{\text{LR}}{=}\lim_{x\to1}\frac{1/(2\sqrt{x})}{2x}=\lim_{x\to1}\frac{1}{4x\sqrt{x}}=\frac{1}{4}.$$

11. The limit is of the form $\dfrac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{\sqrt{x}-1}{x^2-1}\overset{\text{LR}}{=}\lim_{x\to\infty}\frac{1/(2\sqrt{x})}{2x}=\lim_{x\to\infty}\frac{1}{4x\sqrt{x}}=0.$$

12. $\sqrt{x}$ is not defined for negative values of x, so this limit is not defined and cannot be evaluated by any method.

13. The limit is of the form $\dfrac{\infty}{\infty}$. To find the derivative of $\ln(\ln x)$, you'll need the Chain Rule with $f(x) = \ln(x)$ and $g(x) = \ln(x)$.

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{x^2+1} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{\dfrac{1}{\ln x}\cdot\dfrac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{2x^2 \ln x} = 0.$$

14. The limit is of the form $\dfrac{0}{\infty}$, so L'Hôpital's Rule cannot be applied.

15. The limit is of the form $\dfrac{\infty}{\infty}$.

$$\lim_{x\to-\infty} \frac{e^{-x}}{x^2} \overset{\text{LR}}{=} \lim_{x\to-\infty} \frac{-e^{-x}}{2x} \overset{\text{LR}}{=} \lim_{x\to-\infty} \frac{e^{-x}}{2} \text{ DNE } (+\infty).$$

7.3 Asymptotes and Limits at Infinity, III

We saw that $\lim_{x \to \infty} \frac{x^2}{e^x} = 0$, which meant that e^x grows much faster than x^2 as $x \to \infty$. There is nothing special about the exponent of "2," and in fact, we have:

$$\lim_{x \to \infty} \frac{x^n}{e^x} = 0, \quad n > 0.$$

It is very important to realize that this faster growth is as $x \to \infty$. To see why, go to desmos.com. We will look at the horizontal asymptotes of $f(x) = \frac{x}{x + e^x}$.

Looking as $x \to \infty$, we see that this limit is of the form $\frac{\infty}{\infty}$, and so we may apply L'Hôpital's Rule.

$$\lim_{x \to \infty} \frac{x}{x + e^x} \stackrel{\text{LR}}{=} \lim_{x \to \infty} \frac{1}{1 + e^x} = 0.$$

We can observe this visually on the graph. It also makes sense, since e^x grows faster than x, and the e^x is in the denominator.

Looking at $x \to -\infty$, we see that this limit is of the form $\frac{-\infty}{-\infty}$. This is because $\lim_{x \to -\infty} e^x = 0$, meaning the x is the dominant term in the denominator. So for x far to the left,

$$\frac{x}{x + e^x} \approx \frac{x}{x} = 1.$$

We can also see this using L'Hôpital's Rule:

$$\lim_{x \to -\infty} \frac{x}{x + e^x} \stackrel{\text{LR}}{=} \lim_{x \to -\infty} \frac{1}{1 + e^x} = \frac{1}{1 + 0} = 1.$$

Thus, as we look to the left, we see a horizontal asymptote at $y = 1$. We summarize these observations.

Suppose $n > 0$. Then:

1. as $x \to \infty$, e^x dominates x^n, and
2. as $x \to -\infty$, x^n dominates e^x (if x^n is well-defined).

By well-defined, we mean that x^n exists. For example, when $x < 0$, $x^{1/2} = \sqrt{x}$ is not defined, but $x^{1/3} = \sqrt[3]{x}$ is defined.

We could undertake a similar investigation with $\ln x$, but suffice it to say that

$$\lim_{x\to\infty} \frac{\ln x}{x^n} = 0, \quad n > 0.$$

Exponentials and Logarithms

We've compared powers of x to e^x and $\ln x$, but what about other bases? Remember, $\ln x = \log_e x$. For example, what about

$$\lim_{x\to\infty} \frac{x^2}{2^x}?$$

If we want to use L'Hôpital's Rule, we need to be able to take the derivative of $h(x) = 2^x$. To do this, we observe that since e^x and $\ln x$ are inverse functions, then $2 = e^{\ln 2}$. This means that

$$2^x = \left(e^{\ln 2}\right)^x = e^{(\ln 2)x}.$$

So we can write $h(x)$ as $f(g(x))$, where $f(x) = e^x$ and $g(x) = (\ln 2)x$. Then $f'(x) = e^x$ and $g'(x) = \ln 2$. Using the chain rule, we have

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= e^{g(x)} \ln 2 \\ &= e^{(\ln 2)x} \ln 2 \\ &= 2^x \ln 2. \end{aligned}$$

Now we can use this in L'Hôpital's Rule.

$$\lim_{x\to\infty} \frac{x^2}{2^x} \overset{\text{LR}}{=} \frac{2x}{2^x \ln 2} \overset{\text{LR}}{=} \frac{2}{2^x(\ln 2)(\ln 2)} = 0.$$

We can see this by looking at the graphs in desmos. So as long as the base is larger than 1 (or else the exponential function is *decreasing*), these quotients behave similarly. Since the same calculations work for any base:

Suppose $b > 1$ and $n > 0$. Then:

1. $\dfrac{d}{dx} b^x = b^x \ln b$,
2. $\displaystyle\lim_{x\to\infty} \frac{x^n}{b^x} = 0$,
3. as $x \to \infty$, b^x dominates x^n, and
4. as $x \to -\infty$, x^n dominates b^x (if x^n is well-defined).

Logarithms can also occur in other bases, as in

$$\lim_{x\to\infty} \frac{\log_2 x}{\sqrt{x}}.$$

We can tackle limits like these by remembering the change of base formula for logarithms:

> Suppose $b,\ c > 0$. Then for *any* other base $a > 1$,
> $$\log_b c = \frac{\log_a c}{\log_a b}.$$

Why this helps is that we know all about the base e. So we can use $a = e$ in the change of base formula, giving

> Suppose $b,\ c > 0$. Then
> $$\log_b c = \frac{\ln c}{\ln b}.$$

Let's use this to take the derivative of $p(x) = \log_2 x$. Remember that $\ln 2$ is just a constant.

$$\begin{aligned} p(x) &= \log_2 x \\ &= \frac{\ln x}{\ln 2} \\ p'(x) &= \frac{1}{x \ln 2} \end{aligned}$$

Now we can use this to evaluate our limit. Note that it is of the form $\dfrac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{\log_2 x}{\sqrt{x}} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{\dfrac{1}{x\ln 2}}{\dfrac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{1}{x\ln 2} \cdot \frac{2\sqrt{x}}{1} = \lim_{x\to\infty} \frac{2}{\sqrt{x}\ln 2} = 0.$$

The same calculations work for any base, so:

> Suppose $b > 1$ and $n > 0$. Then:
>
> 1. $\dfrac{d}{dx} \log_b x = \dfrac{1}{x \ln b,}$
> 2. $\displaystyle\lim_{x\to\infty} \frac{\log_b x}{x^n} = 0$, and
> 3. as $x \to \infty$, x^n dominates $\log_b x$.

L'Hôpital's Rule Disguised

Now we'll look at some example of limits which, although *not* of the form that allows us to apply L'Hôpital's Rule, they can be rewritten so L'Hôpital's Rule can be applied. The idea is similar to when we rewrote $\dfrac{1}{x^4}$ as x^{-4} so we could use the Power Rule instead of the Quotient Rule. Best to start with an example.

Find $\lim_{x\to\infty}(x+1)e^{-3x}$. The difficulty with limits like these is that one part blows up, and the other goes to 0. Here, $\lim_{x\to\infty}(x+1)$ DNE $(+\infty)$ and $\lim_{x\to\infty} e^{-3x} = 0$. We need to see what one "wins." We call this type of limit "$0\cdot\infty$" or "$\infty\cdot 0$."

But isn't 0 times anything equal to 0? Yes, if that anything is a *number.* But ∞ is *not* a number, but represents numbers getting larger than larger. Below are three limits of the form $\infty\cdot 0$.

$$\begin{aligned}
\lim_{x\to\infty} x\cdot\frac{1}{x^2} &= \lim_{x\to\infty}\frac{1}{x} = 0,\\
\lim_{x\to\infty} x\cdot\frac{1}{x} &= \lim_{x\to\infty} 1 = 1,\\
\lim_{x\to\infty} x^2\cdot\frac{1}{x} &= \lim_{x\to\infty} x \text{ DNE } (+\infty).
\end{aligned}$$

So a limit of the form $\infty\cdot 0$ can be 0, a nonzero number, or might not even exist. So you can't automatically say it's 0.

So what can we do? Remember, a negative exponent lets us move an expression to the denominator. So

$$\lim_{x\to\infty}(x+1)e^{-3x} = \lim_{x\to\infty}\frac{x+1}{e^{3x}}.$$

Now it is of the form $\frac{\infty}{\infty}$, so L'Hôpital's Rule can be applied.

$$\lim_{x\to\infty}\frac{x+1}{e^{3x}} \overset{\text{LR}}{=} \lim_{x\to\infty}\frac{1}{3e^{3x}} = 0.$$

Let's try another: $\lim_{x\to 0^+} x\ln x$. As you can see in the desmos notebook, this limit should be 0, but we'll use calculus to show it. This limit is of the form $0\cdot\infty$. Note that 0^+ is needed since $\ln x$ is not defined for $x\le 0$. There's no negative exponent here, so we have two options:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{x}{1/\ln x} = \lim_{x\to 0^+}\frac{\ln x}{1/x}.$$

The last two limits are of the form $\frac{\infty}{\infty}$. Which one should we try? Let's take them one at a time. For the first, we'll need to take the derivative of $(\ln x)^{-1}$, so let's use the Chain Rule first before applying L'Hôpital's Rule. We write $h(x) = (\ln x)^{-1}$ as $f(g(x))$, where $f(x) = \frac{1}{x} = x^{-1}$ and $g(x) = \ln x$. So $f'(x) = -x^{-2}$ and $g'(x) = \frac{1}{x}$.

$$\begin{aligned}
h(x) &= f(g(x))\\
h'(x) &= f'(g(x))g'(x)\\
&= -(g(x))^{-2}\cdot\frac{1}{x}\\
&= -(\ln x)^{-2}\cdot\frac{1}{x}\\
&= -\frac{1}{x(\ln x)^2}
\end{aligned}$$

Therefore,

$$\lim_{x\to 0^+} \frac{x}{1/\ln x} \stackrel{\text{LR}}{=} \frac{1}{-\dfrac{1}{x(\ln x)^2}}$$
$$= \lim_{x\to 0+} -x(\ln x)^2$$

The problem? This limit is *still* of the form $0 \cdot \infty$. And instead of a $\ln x$, we have a $(\ln x)^2$, which seems to make the problem worse. So let's try the other way.

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} \stackrel{\text{LR}}{=} \lim_{x\to 0^+} \frac{\dfrac{1}{x}}{-\dfrac{1}{x^2}} = \lim_{x\to 0^+} \frac{1}{x} \cdot \frac{-x^2}{1} = \lim_{x\to 0^+} (-x) = 0.$$

You'll notice that the first way only made the problem harder, but the second way wasn't too difficult. So how do you decide? The rule of thumb is that if you have a limit like this involving $\ln x$, just leave the $\ln x$ where it is, and move the other term.

Homework

1. Find the derivatives of the following functions.
   (a) $h(x) = 5^x$
   (b) $h(x) = \log_3 x$
   (c) $h(x) = \log_2(x^2 + 1)$
   (d) $h(x) = 4^{3x+1}$
   (e) $h(x) = \log_5(x^2 5^x)$
2. Find the following limits.
   (a) $\lim_{x\to\infty} e^{-3x} \ln x$
   (b) $\lim_{x\to\infty} \frac{4^x}{3^x}$
   (c) $\lim_{x\to-\infty} \frac{4^x}{3^x}$
   (d) $\lim_{x\to-\infty} e^{2x} x^2$
   (e) $\lim_{x\to 0^+} x^2 \ln x$

Solutions

1. (a) $h'(x) = 5^x \ln 5$

   (b) $h'(x) = \dfrac{1}{x \ln 3}$

   (c) Use the Chain Rule with $f(x) = \log_2 x$ and $g(x) = x^2 + 1$. Then $f'(x) = \dfrac{1}{x \ln 2}$ and $g'(x) = 2x$. Therefore,

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= \frac{1}{g(x)\ln 2} \cdot 2x \\ &= \frac{2x}{(x^2+1)\ln 2} \end{aligned}$$

   (d) Use the Chain Rule with $f(x) = 4^x$ and $g(x) = 3x+1$. Then $f'(x) = 4^x \ln 4$ and $g'(x) = 3$.

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= 4^{g(x)}(\ln 4) \cdot 3 \\ &= 3 \cdot 4^{3x+1} \ln 4 \end{aligned}$$

   (e) Use rules of logarithms to simplify first.

$$f(x) = \log_5(x^2 5^x) = \log_5(x^2) + \log_5(5^x) = 2\log_5(x) + x.$$

   Then

$$h'(x) = \frac{2}{x \ln 5} + 1.$$

2. (a) This limit is of the form $0 \cdot \infty$. Rewrite and use L'Hôpital's Rule.

$$\lim_{x\to\infty} e^{-3x} \ln x = \lim_{x\to\infty} \frac{\ln x}{e^{3x}} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{1/x}{3e^{3x}} = \lim_{x\to\infty} \frac{1}{3xe^{3x}} = 0.$$

   (b) This limit is of the form $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule, we get

$$\lim_{x\to\infty} \frac{4^x}{3^x} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{4^x \ln 4}{3^x \ln 3}.$$

   This is still of the form $\dfrac{\infty}{\infty}$, and using L'Hôpital's Rule again will not help. But using rules of exponents,

$$\frac{4^x}{3^x} = \left(\frac{4}{3}\right)^x.$$

   So

$$\lim_{x\to\infty} \left(\frac{4}{3}\right)^x \text{ DNE } (+\infty),$$

   since we are taking a number which is greater than 1 to larger and larger powers.

(c) This limit is of the form $\frac{0}{0}$. Using L'Hôpital's Rule, we get

$$\lim_{x\to-\infty}\frac{4^x}{3^x}\stackrel{\text{LR}}{=}\lim_{x\to-\infty}\frac{4^x\ln 4}{3^x\ln 3}.$$

This is still of the form $\frac{0}{0}$, and using L'Hôpital's Rule again will not help. But using rules of exponents,

$$\frac{4^x}{3^x}=\left(\frac{4}{3}\right)^x.$$

So

$$\lim_{x\to-\infty}\left(\frac{4}{3}\right)^x=0,$$

since we are taking a number which is greater than 1 to more negative powers.

(d) This limit is of the form $0\cdot\infty$. So we rewrite as

$$\lim_{x\to-\infty}\frac{x^2}{e^{-2x}}\stackrel{\text{LR}}{=}\lim_{x\to-\infty}\frac{2x}{-2e^{-2x}}\stackrel{\text{LR}}{=}\lim_{x\to\infty}\frac{2}{4e^{-2x}}=0.$$

(e) This limit is of the form $0\cdot\infty$. We move the x^2 term.

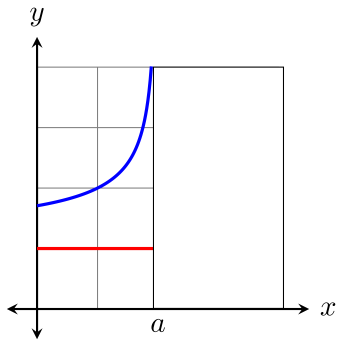
$$\lim_{x\to0^+}\frac{\ln x}{x^{-2}}\stackrel{\text{LR}}{=}\lim_{x\to0^+}\frac{1/x}{-2x^{-3}}=\lim_{x\to0^+}\frac{1/x}{-2/x^3}=\lim_{x\to0^+}\frac{1}{x}\cdot\frac{-x^3}{2}=\lim_{x\to0^+}\frac{-x^2}{2}=0.$$

7.4 Summary of Limits in Calculus

We used limits at various different points so far. They were necessary to define the derivative. We also used them to define what it means to be continuous or discontinuous, which includes essential and removable discontinuites. Then we used limits to describe the behavior of the graph of a function at horizontal and vertical asymptotes. Here, we summarize all the important points.

The limit of $f(x)$ as x approaches a from the left:

$$\lim_{x\to a^-} f(x)$$

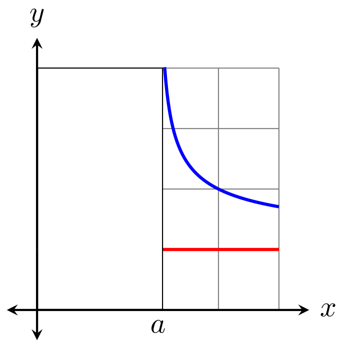


Used for:

1. Determining continuity,
2. Describing discontinuities,
3. Describing behavior at an asymptote.

The limit of $f(x)$ as x approaches a from the right:

$$\lim_{x\to a^+} f(x)$$



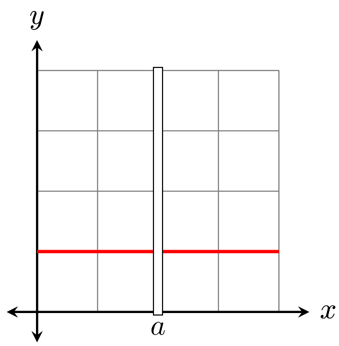
Used for:

1. Determining continuity,
2. Describing discontinuities,
3. Describing behavior at an asymptote.

The limit of $f(x)$ as x approaches a : $\lim_{x\to a} f(x)$.

Only exists if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x).$$



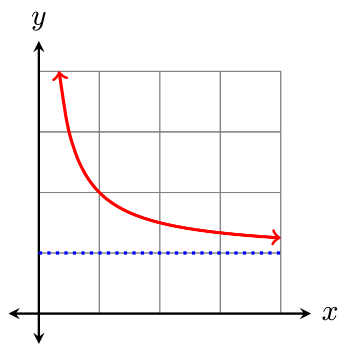
Used for:

1. Determining continuity,
2. Describing discontinuities.

The limit of $f(x)$ as x approaches infinity:

$$\lim_{x\to\infty} f(x)$$

.



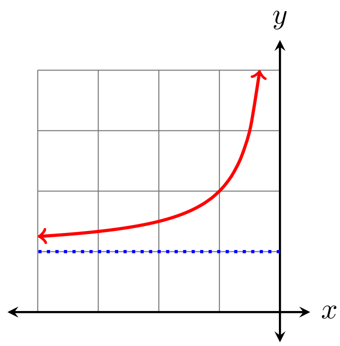
Used for:

1. Determining a horizontal asymptote to the right.

The limit of $f(x)$ as x approaches negative infinity:

$$\lim_{x\to-\infty} f(x)$$

.

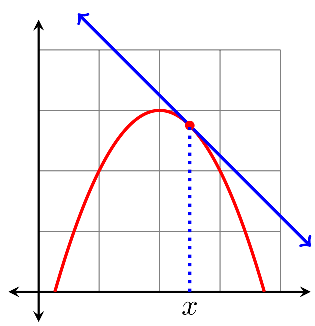


Used for:

1. Determining a horizontal asymptote to the left.

The derivative of $f(x)$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$



Used for:

1. Determining the slope of the tangent line.

Note that the h *must* cancel somehow.

Evaluating Limits

There are many ways to evaluate limits. Let's summarize the ones we encounter most often.

In ALL cases where the limit DNE (does not exist), you must do additional work to see if the limit DNE ($-\infty$), DNE ($+\infty$), or just DNE (for example, when a function approaches an asymptote in opposite directions).

1. Sometimes you can just plug in. The commonly occurs when using the definition of the derivative. You can't plug in right away, since you get $\frac{0}{0}$. But once the h cancels out, you can usually just plug in.

2. When the limit involves a quotient, there are two primary methods.

   (a) If the limit is the limit of $x \to \pm\infty$ of a rational function $f(x)$ (numerator and denominator are polynomials), and if N is the degree of the numerator and D is the degree of the denominator, then:

      i. If $N < D$, then $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0$;

      ii. If $N = D$, then $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x)$ is the ratio of the leading coefficients of the numerator and the denominator;

      iii. If $N > D$, then $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ DNE.

   (b) If the quotient does *not* involve a rational function, then evaluate using the following chart, where LR stands for L'Hôpital's Rule. Again, DNE can also mean DNE ($-\infty$) or DNE ($+\infty$), but more work usually has to be done to determine if one of these applies. Here, "$\neq 0$" means not 0, but *also* not $\pm\infty$.

NUM / DEN	0	$\neq 0$	$\pm\infty$
0	LR	DNE	DNE
$\neq 0$	0	PLUG IN	DNE
$\pm\infty$	0	0	LR

3. When the limit involves a product, such as $\lim_{x\to a} f(x)g(x)$, where a can be a number or $\pm\infty$, you can often just plug in. If not, use the following chart. Here, the label "$f(x)$" means what $\lim_{x\to a} f(x)$ is, and "$g(x)$" means what $\lim_{x\to a} g(x)$ is. LR means that you have to rewrite the product as a quotient and use L'Hôpital's Rule. Two rules of thumb:

   (a) Move a term with a negative exponent to the denominator;

   (b) Leave a logarithm on the numerator.

   In all other cases, take your best guess. Remember, if you rewrite

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}},$$

   you have to take the derivative of $\frac{1}{g(x)} = (g(x))^{-1}$ using the Chain Rule. Try to determine which way to rewrite which will be easiest.

$g(x)$ \ $f(x)$	0	$\neq 0$	$\pm\infty$
0	0	0	LR
$\neq 0$	0	PLUG IN	DNE
$\pm\infty$	LR	DNE	DNE

4. Some limits we just "know" and do not need to use the charts. They all just "make sense," or we studied them in detail in class. Here, "$n > 0$" means *any* positive number, not just an integer.

   (a) $\lim_{x\to\infty} x^n$ DNE ($+\infty$), where $n > 0$.

   (b) $\lim_{x\to-\infty} x^n$ DNE, where $n > 0$ and x^n is well-defined.

   (c) $\lim_{x\to\infty} \frac{1}{x^n} = 0$, where $n > 0$.

   (d) $\lim_{x\to-\infty} \frac{1}{x^n} = 0$, where $n > 0$ and x^n is well-defined.

   (e) $\lim_{x\to 0^+} \frac{1}{x^n}$ DNE ($+\infty$), where $n > 0$.

   (f) $\lim_{x\to 0^-} \frac{1}{x^n}$ DNE, where $n > 0$ and x^n is well-defined.

   (g) $\lim_{x\to\infty} b^x$ DNE ($+\infty$), where $b > 1$.

   (h) $\lim_{x\to-\infty} b^x = 0$, where $b > 1$.

   (i) $\lim_{x\to\infty} \ln x$ DNE ($+\infty$).

   (j) $\lim_{x\to 0^+} \ln x$ DNE ($-\infty$).

   (k) $\lim_{x\to\infty} \frac{x^n}{b^x} = 0$, where $n > 0$ and $b > 1$.

   (l) $\lim_{x\to\infty} \frac{\log_b x}{x^n} = 0$, where $n > 0$ and $b > 1$.

In-Class Practice/Homework

When a limit DNE, determine whether it is DNE ($+\infty$), DNE ($-\infty$), or DNE.

1. $\displaystyle\lim_{x\to-\infty} \frac{1-2x^2}{3x^2-4x-1}$
2. $\displaystyle\lim_{x\to2^-} \frac{x^2+1}{x^2-4}$
3. $\displaystyle\lim_{x\to2^+} \frac{x^2+1}{x^2-4}$
4. $\displaystyle\lim_{x\to2} \frac{x^2+1}{x^2-4}$
5. $\displaystyle\lim_{x\to-\infty} \frac{x^4-2x-6}{x-4}$
6. $\displaystyle\lim_{x\to\infty} e^{-x}\ln x$
7. $\displaystyle\lim_{x\to0^+} e^{x}\sin(x)$
8. $\displaystyle\lim_{x\to0^-} \frac{x^2}{\sin(x)}$
9. $\displaystyle\lim_{x\to0^-} \frac{x^2+1}{\sin(x)}$
10. $\displaystyle\lim_{x\to-\infty} x^5\ln x$
11. $\displaystyle\lim_{x\to\infty} x^5\ln x$
12. $\displaystyle\lim_{x\to0^+} x^5\ln x$
13. $\displaystyle\lim_{x\to\infty} \frac{e^x}{x}.$
14. $\displaystyle\lim_{x\to\infty} \frac{2^x}{e^x}.$
15. $\displaystyle\lim_{x\to0^-} \frac{\sin(x)}{\tan(x)}.$

Solutions

1. This is a rational function with $N = 2$ and $D = 2$. Therefore, the limit is the ratio of the leading coefficients, $-\frac{2}{3}$.

2. The numerator approaches 5, while the denominator approaches 0. Thus, this limit DNE. Since the numerator is positive and the denominator is negative – plug in 1.99 to get $1.99^2 - 4 \approx -0.04$ – the limit is DNE $(-\infty)$.

3. The numerator approaches 5, while the denominator approaches 0. Thus, this limit DNE. Since the numerator is positive and the denominator is positive – plug in 2.01 to get $2.01^2 - 4 \approx 0.04$ – the limit is DNE $(+\infty)$.

4. Based on the previous two answers, this limit is DNE, since the graph approaches the asymptote $x = 2$ from opposite directions.

5. $N = 4$ and $D = 1$, so since $N > D$, this limit DNE. To see if it is DNE $(+\infty)$ or DNE $(-\infty)$, we look at the highest degrees of the numerator and denominator here as $x \to -\infty$. So

$$\frac{x^4 - 2x - 6}{x - 4} \approx \frac{x^4}{x} = x^3.$$

Since the cube of a negative number is negative, this limit is DNE $(-\infty)$.

6. This limit is of the form $0 \cdot \infty$. We move the negative exponent to the denominator and use L'Hôpital's Rule.

$$\lim_{x\to\infty} e^{-x} \ln x = \lim_{x\to\infty} \frac{\ln x}{e^x} \stackrel{\text{LR}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} = \lim_{x\to\infty} \frac{1}{xe^x} = 0.$$

7. Here, we can just plug in. $\lim_{x\to 0^+} e^x \sin(x) = e^0 \sin(0) = 0.$

8. This limit is of the form $\frac{0}{0}$, so we can apply L'Hôpital's Rule.

$$\lim_{x\to 0^-} \frac{x^2}{\sin(x)} \stackrel{\text{LR}}{=} \lim_{x\to 0^-} \frac{2x}{\cos(x)} = \frac{2 \cdot 0}{\cos(0)} = 0.$$

9. The numerator goes to 1 and the denominator goes to 0, so this limit DNE. To see if it is DNE $(+\infty)$ or DNE $(-\infty)$, note that 1 is positive. Looking at a graph of $y = \sin(x)$, we see that coming to 0 from the left, $\sin(x)$ is negative. Since $\frac{+}{-} = -$, this limit is DNE $(-\infty)$.

10. This limit is undefined, since $\ln x$ is not defined for negative numbers. It is important to point out that "undefined" is different that "DNE." To say that a limit is undefined is to say that you can't even evaluate it because the x-values don't make sense for the functions in the limit. To say that a limit is DNE means the limit makes sense as far as the x-values are concerned, but there is no limiting value.

11. This limit is of the form $+\infty \cdot +\infty$, so this limit DNE $(+\infty)$.

12. This limit is of the form $0 \cdot -\infty$, so we must rewrite and use L'Hôpital's Rule. We keep the logarithm on the numerator.

$$\lim_{x\to 0^+} x^5 \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-5}} \stackrel{\text{LR}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-5x^{-6}} \cdot \frac{x^6}{x^6} = \lim_{x\to 0^+} \frac{x^5}{-5} = 0.$$

13. We know that this limit DNE since e^x dominates x. As $x \to \infty$, this limit is of the form $\frac{+}{+} = +$, so the limit is DNE $(+\infty)$.

14. This limit is of the form $\frac{\infty}{\infty}$, but using L'Hôpital's Rule won't help because the exponential functions will still remain. But we can rewrite as $\lim_{x\to\infty} \left(\frac{2}{e}\right)^x = 0$. Since $2 < e$, we are taking higher and higher powers of a number less than 1, so this limit is 0.

15. This limit is of the form $\frac{0}{0}$, so it looks like a L'Hôpital's Rule problem. But

$$\frac{\sin(x)}{\tan(x)} = \frac{\sin(x)}{\frac{\sin(x)}{\cos(x)}} = \frac{\sin(x)}{1} \cdot \frac{\cos(x)}{\sin(x)} = \cos(x),$$

so this limit is

$$\lim_{x\to 0^-} \cos(x) = 1.$$

Chapter 8

Further Applications

8.1 Tangents to Curves

Up to this point, we used calculus and derivatives to look at properties of graphs of functions. But what about curves which are *not* graphs of functions? That is, curves that do not pass the vertical line test?

A simple example is a circle, such as $x^2 + y^2 = 9$. There are two ways we can approach this. First,

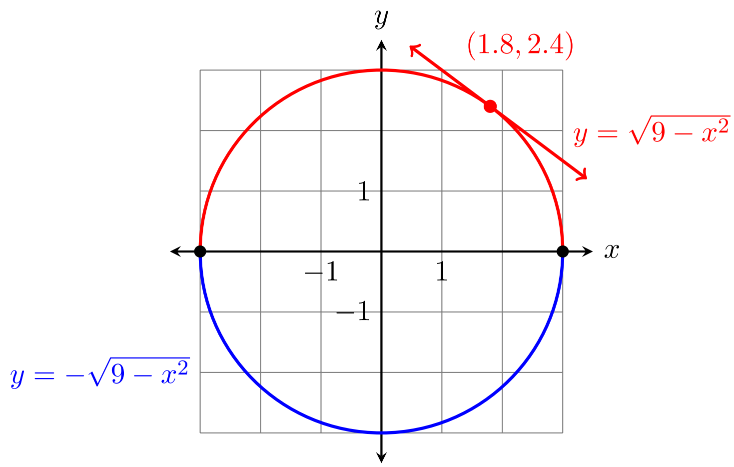


Figure 8.1: Describing a circle using two functions.

we can solve for y and look at the top and bottom halves of the circle separately. Each half is the graph of a function, as shown in Figure 8.1. Let's find an equation for the tangent line at $(1.8, 2.4)$. We'll use the Chain Rule with $h(x) = \sqrt{9 - x^2}$, using $f(x) = \sqrt{x}$ and $g(x) = 9 - x^2$, and so $f'(x) = \dfrac{1}{2\sqrt{x}}$ and $g'(x) = -2x$.

$$\begin{aligned}
h(x) &= \sqrt{9 - x^2} \\
h'(x) &= f'(g(x))g'(x) \\
&= \frac{1}{2\sqrt{g(x)}} \cdot (-2x) \\
&= -\frac{2x}{2\sqrt{9 - x^2}} \\
&= -\frac{x}{\sqrt{9 - x^2}}
\end{aligned}$$

Plugging in $x = 1.8$, we get $h'(1.8) = -0.75$. To find an equation for the tangent line, we use the point-slope formula.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 2.4 &= -0.75(x - 1.8) \\
y - 2.4 &= -0.75x + 1.35 \\
y &= -0.75x + 3.75 \\
y &= -\frac{1}{4}x + \frac{15}{4}
\end{aligned}$$

What made this problem a lot of work was having to work with the square root. It turns out there is an easier way using a method called **implicit differentiation.** If we start with $y = f(x)$ and take the derivative, we get $\frac{dy}{dx} = f'(x)$. In other words, differentiate both sides:

$$\begin{aligned} y &= f(x) \\ \frac{d}{dx}y &= \frac{d}{dx}f(x) \\ \frac{dy}{dx} &= f'(x) \end{aligned}$$

Now if we don't have the graph of a function, we cannot use $f'(x)$. So when we perform implicit differentiation, we use the notation $\frac{dy}{dx}$. Let's begin with our original equation and differentiate both sides:

$$\begin{aligned} x^2 + y^2 &= 9 \\ \frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= \frac{d}{dx}9 \\ 2x + \frac{d}{dx}y^2 &= 0 \end{aligned}$$

Two of the terms are easy, but what do we do with $\frac{d}{dx}y^2$? Here, we need the Chain Rule. Let's call $p(x) = y^2$, so $f(x) = x^2$ and $g(x) = y$. Then $f'(x) = 2x$ and $g'(x) = \frac{dy}{dx}$. So

$$\begin{aligned} p(x) &= y^2 \\ p'(x) &= \frac{d}{dx}y^2 \\ &= f'(g(x))g'(x) \\ &= f'(y)\frac{dy}{dx} \\ &= 2y\frac{dy}{dx} \end{aligned}$$

Let's look at this for a moment:

$$\frac{d}{dx}y^2 = 2y\frac{dy}{dx}.$$

It looks like we differentiate y^2 where y is the variable, and then multiply by $\frac{dy}{dx}$. Looking back at the Chain Rule, that's *exactly* what we did. So

$$\frac{d}{dx}\sin(y) = \cos(y)\frac{dy}{dx}, \quad \frac{d}{dx}e^y = e^y\frac{dy}{dx}, \quad \frac{d}{dx}\ln y = \frac{1}{y}\frac{dy}{dx},$$

and so on. Once we observe this pattern, we can skip the Chain Rule each time. But it's important to see where the pattern comes from.

Going back, let's substitute $\frac{d}{dx}y^2 = 2y\frac{dy}{dx}$ back in and solve for $\frac{dy}{dx}$.

$$\begin{aligned} 2x + \frac{d}{dx}y^2 &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ &= -\frac{x}{y} \end{aligned}$$

This is much simpler algebraically. Also notice that it is the same result, since because $y = \sqrt{9 - x^2}$, then

$$-\frac{x}{y} = -\frac{x}{\sqrt{9 - x^2}}.$$

Also, the slope at $(1.8, 2.4)$ is easier to find:

$$-\frac{x}{y} = -\frac{1.8}{2.4} = -\frac{3}{4}.$$

So often, implicit differentiation can be much easier to use. Sometimes, it is *impossible* to solve for y, as we will see later, and then there is no other viable option.

Example 1

Consider the ellipse defined by $x^2 + xy + y^2 = 12$, which you can see graphed at desmos.com. Give equations for the horizontal and vertical tangents to this curve.

Implicit differentiation is very helpful here, since otherwise, we would need to use the quadratic formula to solve for y, which would make the algebra really messy. So let's first find $\frac{dy}{dx}$. We will use the fact that $\frac{d}{dx}y^2 = 2y\frac{dy}{dx}$ from the last problem.

$$\begin{aligned} x^2 + xy + y^2 &= 12 \\ \frac{d}{dx}x^2 + \frac{d}{dx}xy + \frac{d}{dx}y^2 &= \frac{d}{dx}12 \\ 2x + \frac{d}{dx}xy + 2y\frac{dy}{dx} &= 0 \end{aligned}$$

How do we handle $\frac{d}{dx}xy$? Here, we need the product rule. Since we're using the notation $\frac{dy}{dx}$, let's rewrite the Product Rule first. Suppose $h(x) = f(x)g(x)$.

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ \frac{d}{dx}h(x) &= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) \end{aligned}$$

Now put $h(x) = xy$, with $f(x) = x$ and $g(x) = y$, so that $f'(x) = 1$ and $g'(x) = \frac{dy}{dx}$. Then

$$\begin{aligned} \frac{d}{dx}h(x) &= \frac{d}{dx}xy \\ &= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) \\ &= x \cdot \frac{dy}{dx} + y \cdot 1 \\ &= x\frac{dy}{dx} + y \end{aligned}$$

Now let's substitute back in and solve for $\frac{dy}{dx}$.

$$\begin{aligned} 2x + \frac{d}{dx}xy + 2y\frac{dy}{dx} &= 0 \\ 2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} &= 0 \\ x\frac{dy}{dx} + 2y\frac{dy}{dx} &= -(2x + y) \\ (x + 2y)\frac{dy}{dx} &= -(2x + y) \\ \frac{dy}{dx} &= \frac{-(2x + y)}{x + 2y} \end{aligned}$$

Let's see how we use this to find horizontal and vertical tangents. For a horizontal tangent, we need $\frac{dy}{dx} = 0$, and so the numerator of $\frac{dy}{dx}$ must be 0. Then

$$\begin{aligned} -(2x+y) &= 0 \\ 2x+y &= 0 \\ y &= -2x \end{aligned}$$

What does this mean? In desmos, make sure that the ellipse and the equation $y = -2x$ are selected. You can see immediately that they intersect at the points where there are horizontal tangents. So these points must satisfy the equations:

$$\begin{aligned} x^2+xy+y^2 &= 12 \\ y &= -2x \end{aligned}$$

So to find the points, we can substitute in and solve.

$$\begin{aligned} x^2+xy+y^2 &= 12 \\ x^2+x(-2x)+(-2x)^2 &= 12 \\ x^2-2x^2+4x^2 &= 12 \\ 3x^2 &= 12 \\ x^2 &= 4 \\ x &= -2, +2 \end{aligned}$$

And since $y = -2x$, the two points where there are horizontal tangents are $(-2, 4)$ and $(2, -4)$, as we can see from looking at the graph.

This looks like a *lot* of algebra – and it is. But the important point is that we're not really using any *new* formulas, just applying old formulas in a new situation.

What about the vertical tangents? A vertical tangent has undefined slope. This means the denominator of $\frac{dy}{dx}$ must be 0.

$$\begin{aligned} x+2y &= 0 \\ x &= -2y \\ y &= -\frac{1}{2}x \end{aligned}$$

In desmos, make sure that the ellipse and the equation $y = -\frac{1}{2}x$ are selected. You can see right away that they intersect at the points where there are vertical tangents. So these points must satisfy the equations:

$$\begin{aligned} x^2+xy+y^2 &= 12 \\ x &= -2y \end{aligned}$$

Notice that we use the equation in the form $x = -2y$. This is because to graph, we solve $y = -\frac{1}{2}x$. But to substitute, it's easier not to have to worry about fractions.

$$\begin{aligned} x^2 + xy + y^2 &= 12 \\ (-2y)^2 + (-2y)y + y^2 &= 12 \\ 4y^2 - 2y^2 + y^2 &= 12 \\ 3y^2 &= 12 \\ y^2 &= 4 \\ y &= -2, +2 \end{aligned}$$

Since $x = -2y$, the two points where there are vertical tangents are $(4, -2)$ and $(-4, 2)$. We can see this from looking at the graph.

Example 2 Show that the curve described by the equation $e^{xy} = x + y$ has a horizontal tangent at $(0, 1)$ and a vertical tangent $(1, 0)$.

A graph of this equation is in the desmos notebook. You can see the plausibility of horizontal and vertical tangents at the points $(0, 1)$ and $(1, 0)$, respectively.

We begin by taking derivatives on both sides, as with the other example.

$$\begin{aligned} e^{xy} &= x + y \\ \frac{d}{dx}e^{xy} &= \frac{d}{dx}x + \frac{d}{dx}y \\ \frac{d}{dx}e^{xy} &= 1 + \frac{dy}{dx} \end{aligned}$$

Now we need to deal with the $\frac{d}{dx}e^{xy}$ term. First, we need to use Chain Rule. We use $h(x) = e^{xy}$, so that $f(x) = e^x$ and $g(x) = xy$. Then $f'(x) = e^x$, and from the last example, we have $\frac{d}{dx}g(x) = \frac{d}{dx}xy = x\frac{dy}{dx} + y$. Then

$$\begin{aligned} \frac{d}{dx}e^{xy} &= f'(g(x))g'(x) \\ &= e^{g(x)} \cdot \left(x\frac{dy}{dx} + y\right) \\ &= e^{xy}\left(x\frac{dy}{dx} + y\right) \end{aligned}$$

Now let's substitute back in and solve for $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d}{dx}e^{xy} &= 1 + \frac{dy}{dx} \\ e^{xy}\left(x\frac{dy}{dx} + y\right) &= 1 + \frac{dy}{dx} \\ xe^{xy}\frac{dy}{dx} + ye^{xy} &= 1 + \frac{dy}{dx} \\ xe^{xy}\frac{dy}{dx} - \frac{dy}{dx} &= 1 - ye^{xy} \\ (xe^{xy} - 1)\frac{dy}{dx} &= 1 - ye^{xy} \\ \frac{dy}{dx} &= \frac{1 - ye^{xy}}{xe^{xy} - 1} \end{aligned}$$

This seems like a lot of work, but there is no way to solve this equation for y. So here, implicit differentiation *must* be used.

Now let's see how we use this to answer the question of tangents. Let's find $\frac{dy}{dx}$ at the point $(0, 1)$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{1 - ye^{xy}}{xe^{xy} - 1} \\ &= \frac{1 - 1 \cdot e^{0 \cdot 1}}{0 \cdot e^{0 \cdot 1} - 1} \\ &= \frac{1 - 1}{0 - 1} \\ &= 0.\end{aligned}$$

This means that there is a horizontal tangent at $(0, 1)$. Let's look at what happens at the point $(1, 0)$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{1 - ye^{xy}}{xe^{xy} - 1} \\ &= \frac{1 - 0 \cdot e^{1 \cdot 0}}{1 \cdot e^{1 \cdot 0} - 1} \\ &= \frac{1 - 0}{1 - 1} \\ &= \frac{1}{0}.\end{aligned}$$

This slope is undefined, which means there is a vertical tangent at $(1, 0)$.

Homework

NOTE: Please do not jump straight to the Solutions if you are stuck. Go back and study the Examples, since each problem is similar to one of the Examples. You *will* have a problem on the next Exam of the form "Show that $\frac{dy}{dx} = \ldots$" This means if you don't get the answer right the first time, you'll have to go back and recheck your computations. Practice doing this with the Homework so you can do well on that question.

1. Consider the ellipse $4x^2 + y^2 = 8$. Graph this on desmos.
   (a) Show that $\frac{dy}{dx} = -\frac{4x}{y}$.
   (b) Find an equation of the tangent line at the point $(-1, 2)$. Check you answer by graphing it on desmos as well.
2. Graph the hyperbola $xy = x + y$ on desmos.
   (a) Show that
   $$\frac{dy}{dx} = \frac{1-y}{x-1}.$$
   (b) Find an equation of the tangent line when $x = 2$.
   (c) Find an equation of the tangent line when $x = 0$.
3. Graph the hyperbola $2x^2 - xy - y^2 = 9$ on desmos.
   (a) Show that
   $$\frac{dy}{dx} = \frac{4x-y}{x+2y}.$$
   (b) Use this to find where there are vertical tangents to the hyperbola. Verify this on desmos.

Solutions

1. (a) First, find $\frac{dy}{dx}$.

$$\begin{aligned} 4x^2 + y^2 &= 8 \\ \frac{d}{dx}4x^2 + \frac{d}{dx}y^2 &= \frac{d}{dx}8 \\ 8x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -8x \\ \frac{dy}{dx} &= \frac{-8x}{2y} \\ &= -\frac{4x}{y} \end{aligned}$$

(b) At the point $(-1, 2)$,

$$\begin{aligned} \frac{dy}{dx} &= -\frac{4x}{y} \\ &= -\frac{4(-1)}{2} \\ &= 2. \end{aligned}$$

Using the point-slope equation of a line:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 2(x - (-1)) \\ y - 2 &= 2x + 2 \\ y &= 2x + 4 \end{aligned}$$

2. (a) First, find $\frac{dy}{dx}$.

$$\begin{aligned} xy &= x + y \\ \frac{d}{dx}xy &= \frac{d}{dx}x + \frac{d}{dx}y \\ x\frac{dy}{dx} + y &= 1 + \frac{dy}{dx} \\ x\frac{dy}{dx} - \frac{dy}{dx} &= 1 - y \\ (x-1)\frac{dy}{dx} &= 1 - y \\ \frac{dy}{dx} &= \frac{1-y}{x-1} \end{aligned}$$

(b) Substituting $x = 2$, we have

$$\begin{aligned} 2y &= 2 + y \\ y &= 2 \end{aligned}$$

So the slope of the line is

$$\frac{1-2}{2-1} = -1.$$

Using the point-slope equation of a line:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= -1(x - 2) \\ y - 2 &= -x + 2 \\ y &= -x + 4 \end{aligned}$$

(c) Substituting $x = 0$, we have

$$\begin{aligned} 0 \cdot y &= 0 + y \\ y &= 0 \end{aligned}$$

So the slope of the line is

$$\frac{1-0}{0-1} = -1.$$

Using the point-slope equation of a line:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= -1(x - 0) \\ y &= -x \end{aligned}$$

3. (a) First, find $\frac{dy}{dx}$.

$$\begin{aligned}
2x^2 - xy - y^2 &= 9 \\
\frac{d}{dx}2x^2 - \frac{d}{dx}xy - \frac{d}{dx}y^2 &= \frac{d}{dx}9 \\
4x - x\frac{dy}{dx} - y - 2y\frac{dy}{dx} &= 0 \\
-x\frac{dy}{dx} - 2y\frac{dy}{dx} &= -(4x - y) \\
-(x + 2y)\frac{dy}{dx} &= -(4x - y) \\
\frac{dy}{dx} &= \frac{-(4x - y)}{-(x + 2y)} \\
&= \frac{4x - y}{x + 2y}
\end{aligned}$$

(b) There are vertical tangents when the denominator of $\frac{dy}{dx}$ is 0.

$$\begin{aligned}
x + 2y &= 0 \\
x &= -2y
\end{aligned}$$

It is easier to solve for x here to avoid fractions. Substitute this back in and solve for y.

$$\begin{aligned}
2x^2 - xy - y^2 &= 9 \\
2(-2y)^2 - (-2y)y - y^2 &= 9 \\
8y^2 + 2y^2 - y^2 &= 9 \\
9y^2 &= 9 \\
y &= -1, +1
\end{aligned}$$

We can find x since we know that $x = -2y$. So there are vertical tangents at the points $(2, -1)$ and $(-2, 1)$. This can be visually verified by looking at the graph.

8.2 Inverse Trigonometry I

The most famous pair of inverse functions in calculus is e^x and $\ln x$. We learned a lot about $\ln x$ by reflecting e^x across the line $y = x$. Also, it was very important that when we reflected across the line $y = x$, the graph passed the vertical line test, so we were able to define the *function* $f(x) = \ln x$. Each x corresponded to exactly one y.

To follow along, you will need to visit desmos.com. Trigonometric functions are also very important in calculus. But we can't just reflect along the line $y = x$ and be done with it. Let's see why. We'll start with $\sin(x)$, which you can see by selecting ◯1. Now select ◯2 and ◯3, and you'll see the graph of $\sin(x)$ reflected along the line $y = x$. Notice we literally switch the x and y from $y = \sin(x)$ to $x = \sin(y)$ to see the reflection.

Now select ◯4. You'll notice that $x = \sin(y)$ does *not* pass the vertical line test, and so it is not a function. How can we create a function?

If you select ◯5, you see a small part of the graph of $x = \sin(y)$. This part *does* pass the vertical line test, and it is this part of the curve that we use to define the inverse function, $\arcsin(x)$. Many books write $\sin^{-1}(x)$ for the inverse function, but this is confusing since you might think $\sin^{-1}(x) = \dfrac{1}{\sin(x)}$. When you use $\arcsin(x)$, there is no confusion. Just note this in case you look at online resources.

One big difference here. Since e^x and $\ln x$ are inverse functions, $y = e^x$ means exactly the same thing as $x = \ln y$. They are inverses of each other. But

> If $y = \sin(x)$, then it DOES NOT ALWAYS MEAN THAT $x = \arcsin(y)$.

This fact is what makes working with inverse trigonometric functions challenging. Consider e^x and $\ln x$ again. Using interval notation, the domain of e^x is $(-\infty, \infty)$ and the range is $(0, \infty)$. The domain of $\ln(x)$ is $(0, \infty)$ while the range is $(-\infty, \infty)$. Here, the domain and range just switch.

But that can't happen with $\sin(x)$, because when you reflect across $y = x$, you *don't* get a function. Look back on desmos. Notice that the range of $\sin(x)$, $[-1, 1]$, is the domain of $\arcsin(x)$. But the domain of $\sin(x)$, which is $(-\infty, \infty)$, is *not* the range of $\arcsin(x)$, otherwise the vertical line test would fail. So the range of $\arcsin(x)$ is $[-\pi/2, \pi/2]$, since if the range were made any larger, the graph would fail the vertical line test.

Now select only ◯1 and ◯6. When you deselect ◯1, you'll notice that only one piece of $\sin(x)$ remains. This is called **restricting the domain.** Now select ◯2 and ◯5 again. When you reflect $y = \sin(x)$ with restricted domain, you get a function. So that means:

> If $y = \sin(x)$, and if x is in the restricted domain $[-\pi/2, \pi/2]$, then $x = \arcsin(y)$.

Another way of saying it is this. The domain of the restricted $\sin(x)$, which is $[-\pi/2, \pi/2]$, is the range of $\arcsin(x)$. The range of the restricted $\sin(x)$, which is $[-1, 1]$, is the same as the domain of $\arcsin(x)$.

Also note that this is exactly how we defined $\sqrt{x}$. We had to restrict the domain of $y = x^2$ to $[0, \infty)$ in order to get the inverse function. But we are so familiar with the square root function, we hardly notice. Inverse trigonometric functions are not so familiar.

So, because 0, $\pi/4$, and $-\pi/3$ are all in the range of $\arcsin(x)$, then

$$\arcsin(\sin(0)) = 0, \quad \arcsin(\sin(\pi/4)) = \pi/4, \quad \arcsin(\sin(-\pi/3)) = -\pi/3.$$

But because π and $2\pi/3$ are not in the range of $\arcsin(x)$, then

$$\arcsin(\sin(\pi)) \neq \pi, \quad \arcsin(\sin(2\pi/3)) \neq 2\pi/3.$$

So we need a way to work these out. Who comes to the rescue? The unit circle, of course.

Example 1: $\arcsin(\sin(x))$.

Sometimes it's the case that $\arcsin(\sin(x)) \neq x$. $\arcsin(\sin(2\pi/3)) \neq 2\pi/3$, since $2\pi/3$ is not in the range of $\arcsin(x)$. So how do we go about finding $\arcsin(\sin(2\pi/3))$?

Let's start with a unit circle.

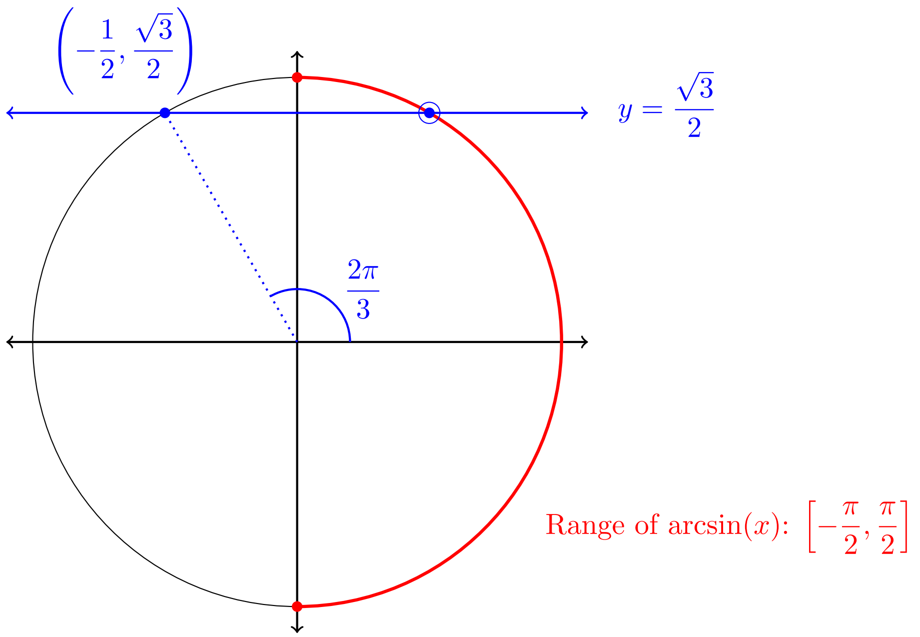


Figure 8.2: Calculating $\arcsin(\sin(2\pi/3))$.

We need to find the appropriate angle in the range of $\arcsin(x)$ whose sine is the same as the sine of $2\pi/3$.

1. Draw a unit circle, and highlight (here in red) the range of $\arcsin(x)$.
2. Since we're looking for $\arcsin(\sin(2\pi/3))$, find the point on the unit circle corresponding to $2\pi/3$ and label the coordinates (blue dot on the left of Figure 8.9).
3. Since $\sin(x)$ is the y-coordinate on the unit circle, draw a horizontal line through this point until in intersects the range of $\arcsin(x)$ (circled blue dot on the right).
4. Find which angle in the range of $\arcsin(x)$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, corresponds to this point on the unit circle.
5. Since $\sin(\pi/3) = \sqrt{3}/2$, then $\arcsin(\sin(2\pi/3)) = \pi/3$.

To summarize, we are essentially asking the question, "What angle in the range of $\arcsin(x)$ has the same sine as $2\pi/3$?"

What about inverses of $\cos(x)$ and $\tan(x)$? We won't go into all the details here, since the basic concept is the same: restrict the domain so that when you reflect the graph, you get the graph of a function – that is, you pass the vertical line test. Select ◯2 and ◯7. When you reflect over $y = x$, you get ◯8. If you select ◯4 again, you'll quickly notice that $x = \cos(y)$ does *not* pass the vertical line test. So, we restrict the domain of $\cos(x)$ to $[0, \pi]$. When you reflect $y = \cos(x)$ with this restricted domain, you get ◯9. See this by selecting ◯2, ◯7, ◯9, and ◯10 only. When you deselect ◯7, you'll see *only* that part of $y \cos(x)$ with domain $[0, \pi]$. Then the inverse relationship is clear. This means that $\arccos(x)$ is the inverse of $y = \cos(x)$ with restricted domain $[0, \pi]$. Thus,

> If $y = \cos(x)$, and if x is in the restricted domain $[0, \pi]$, then $x = \arccos(y)$.

You'll see how to find $\arccos(\cos(x))$ when x does *not* belong to the restricted domain in Example 2. If x is in the restricted domain $[0, \pi]$, then it will always be the case that $\arccos(\cos(x)) = x$.

A similar thing happens with $\tan(x)$. You'll see if you take $y = \tan(x)$ by selecting ◯11, and reflecting about $y = x$ by selecting ◯2 and ◯12, the reflection does not pass the vertical line test. But if we restrict the domain to $(-\pi/2, \pi/2)$ (select ◯2 and ◯13 only) and reflect by selecting ◯14, the graph passes the vertical line test. It is important to note the parentheses: there are vertical asymptotes at $x = -\pi/2$ and $x = \pi/2$, since these points on the unit circle make vertical lines with the origin, and the slope of a vertical line is undefined.

Thus,

> If $y = \tan(x)$, and if x is in the restricted domain $(-\pi/2, \pi/2)$ then $x = \arctan(y)$.

In other words, $\arctan(\tan(x)) = x$ if x is in the restricted domain $(-\pi/2, \pi/2)$. We'll see in Example 3 how to handle the situation if x is not in the restricted domain.

Example 2: $\arccos(\cos(x))$.

Sometimes it's the case that $\arccos(\cos(x)) \neq x$. $\arccos(\cos(5\pi/4)) \neq 5\pi/4$, since $5\pi/4$ is not in the range of $\arccos(x)$. So how do we go about finding $\arccos(\cos(5\pi/4))$?

Again, we start with a unit circle.

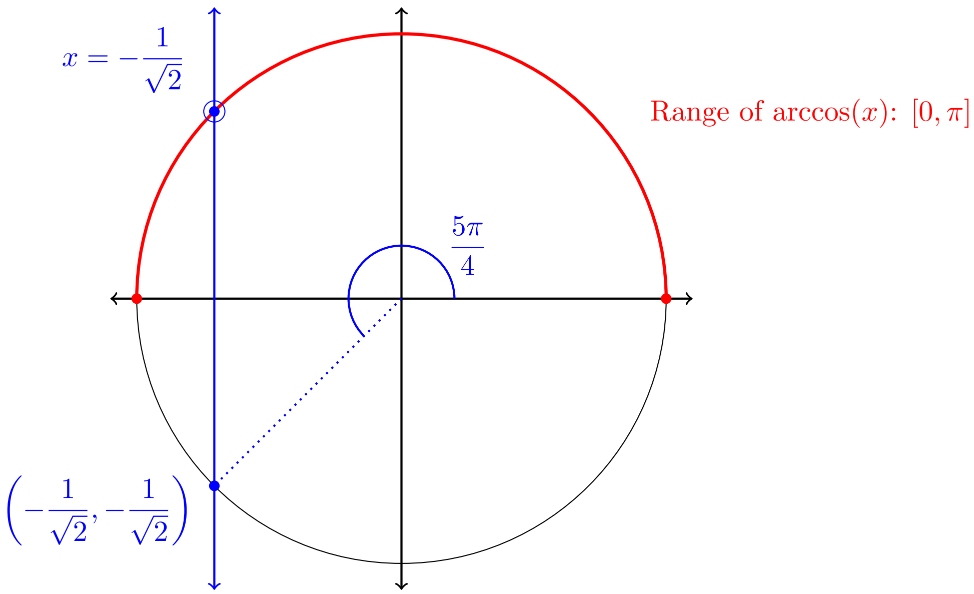


Figure 8.3: Calculating $\arccos(\cos(5\pi/4))$.

We need to find the appropriate angle in the range of $\arccos(x)$ whose cosine is the same as the cosine of $5\pi/4$.

1. Draw a unit circle, and highlight (here in red) the range of $\arccos(x)$.
2. Since we're looking for $\arccos(\cos(5\pi/4))$, find the point on the unit circle corresponding to $5\pi/4$ and label the coordinates (blue dot on the left of Figure 8.3).
3. Since $\cos(x)$ is the x-coordinate on the unit circle, draw a vertical line through this point until in intersects the range of $\arccos(x)$ (circled blue dot on the left).
4. Find which angle in the range of $\arccos(x)$, $[0, \pi]$, corresponds to this point on the unit circle.
5. Since $\cos(3\pi/4) = -1/\sqrt{2}$, then $\arccos(\cos(5\pi/4)) = 3\pi/4$.

To summarize, we are essentially asking the question, "What angle in the range of $\arccos(x)$ has the same cosine as $5\pi/4$?"

Example 3: $\arctan(\tan(x))$.

Sometimes it's the case that $\arctan(\tan(x)) \neq x$. $\arctan(\tan(5\pi/6)) \neq 5\pi/6$, since $5\pi/6$ is not in the range of $\arctan(x)$. So how do we go about finding $\arctan(\tan(5\pi/6))$?

Again, we start with a unit circle.

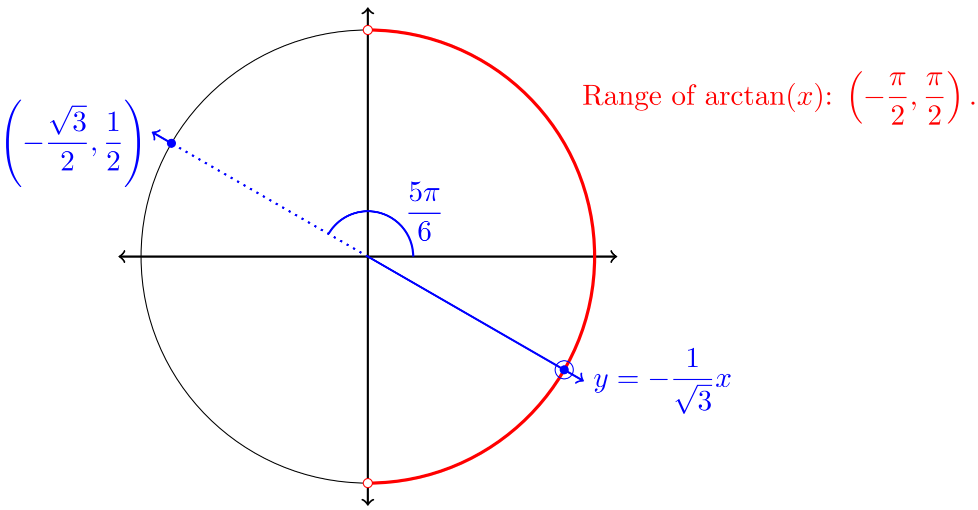


Figure 8.4: Calculating $\arctan(\tan(5\pi/6))$.

We need to find the appropriate angle in the range of $\arctan(x)$ whose tangent is the same as the tangent of $5\pi/6$.

1. Draw a unit circle, and highlight (here in red) the range of $\arctan(x)$.
2. Since we're looking for $\arctan(\tan(5\pi/6))$, find the point on the unit circle corresponding to $5\pi/6$ and label the coordinates (blue dot on the left of Figure 8.4).
3. Now
$$\tan(5\pi/6) = \frac{\sin(5\pi/6)}{\cos(5\pi/6)} = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}}.$$
Since the sine corresponds to the y-coordinate and the cosine corresponds to the x-coordinate, then the tangent corresponds to $\frac{y}{x}$, which is the *slope* of the line through $(-\sqrt{3}/2, 1/2)$ and the origin. Draw this line, and see where it intersects the range of $\arctan(x)$ (circled blue dot on the right).
4. Find which angle in the range of $\arctan(x)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, corresponds to this point on the unit circle.
5. Since $\tan(-\pi/6) = -1/\sqrt{3}$, then $\arctan(\tan(5\pi/6)) = -\pi/6$.

To summarize, we are essentially asking the question, "What angle in the range of $\arctan(x)$ has the same tangent as $5\pi/6$?"

So far, we've looked at how to evaluate $\arcsin(\sin(x))$, $\arccos(\cos(x))$, and $\arctan(\tan(x))$ for all x in the appropriate domain. What about the other way, that is, $\sin(\arcsin(x))$, $\cos(\arccos(x))$, and $\tan(\arctan(x))$? We saw that $\arcsin(\sin(2\pi/3)) \neq 2\pi/3$ because $2\pi/3$ is not in the range of $\arcsin(x)$.

Let's think about what $\sin(\arcsin(x))$ means. The domain of $\arcsin(x)$ is $[-1, 1]$. So x *must* be in the range of $\sin(x)$, because the range of $\sin(x)$ is *also* $[-1, 1]$. This means that $\sin(\arcsin(x)) = x$ for every x in the domain of $\arcsin(x)$, which is $[-1, 1]$. Said another way, any valid x you can plug into $\sin(\arcsin(x))$ will always be in the range of $\sin(x)$, and so $\sin(\arcsin(x)) = x$.

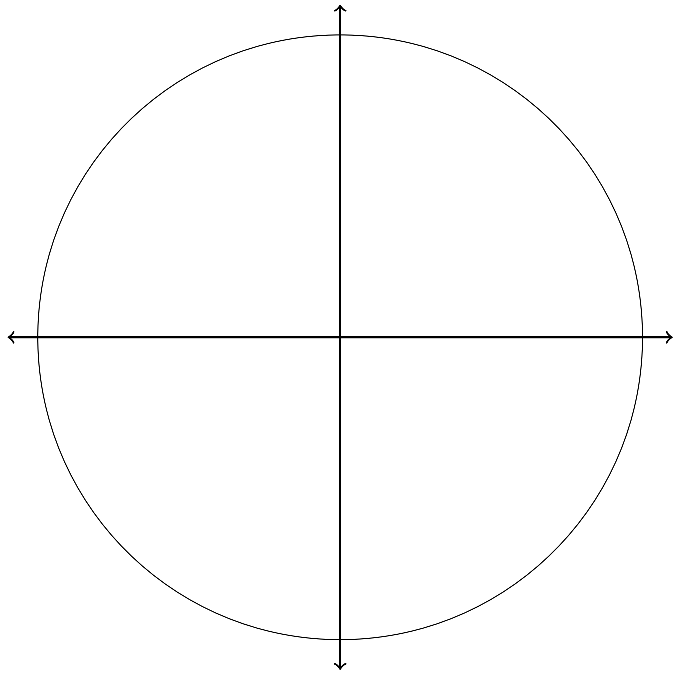
The exact same logic shows that $\cos(\arccos(x)) = x$ and $\tan(\arctan(x)) = x$ for all valid values of x.

The box below summarize all the important points. The tricky parts are 2(a), (b), and (c), where if x is not in the appropriate range, you have to work it out like Examples 1–3 above.

1. (a) For $y = \arcsin(x)$, the domain is $[-1, 1]$, range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
   (b) For $y = \arccos(x)$, the domain is $[-1, 1]$, and the range is $[0, \pi]$.
   (c) For $y = \arctan(x)$, the domain is $(-\infty, \infty)$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
2. (a) $\arcsin(\sin(x)) = x$ for all x in the range of $\arcsin(x)$.
   (b) $\arccos(\cos(x)) = x$ for all x in the range of $\arccos(x)$.
   (c) $\arctan(\tan(x)) = x$ for all x in the range of $\arctan(x)$.
3. (a) $\sin(\arcsin(x)) = x$ for all x in the domain of $\arcsin(x)$.
   (b) $\cos(\arccos(x)) = x$ for all x in the domain of $\arccos(x)$.
   (c) $\tan(\arctan(x)) = x$ for all x in the domain of $\arctan(x)$.

Homework

1. What is a restricted domain, and why is it necessary to define the inverse trigonometric functions?
2. Evaluate $\arccos(\cos(5\pi/3))$.
3. Evaluate $\sin(\arcsin(-\sqrt{3}/2))$.
4. Evaluate $\cos(\arccos(3/2))$.
5. Evaluate $\arctan(\tan(-\pi/4))$.
6. Evaluate $\arcsin(\sin(7\pi/4))$.
7. Evaluate $\tan(\arctan(-100))$.
8. Evaluate $\arccos(\cos(-\pi))$.
9. Evaluate $\arctan(\tan(5\pi/4))$.
10. Evaluate $\arcsin(\sin(4\pi/3))$.



Solutions

1. A restricted domain is when you restrict possible values for x. $\sin(x)$ is defined for all real numbers, but when using it to define $\arcsin(x)$, we restrict the domain to $[-\pi/2, \pi/2]$. We need to do this because when we reflect the graph of $\sin(x)$ across the line $y = x$, the graph does not pass the vertical line test.

2. $\arccos(\cos(5\pi/3)) = \pi/3$, as demonstrated below.

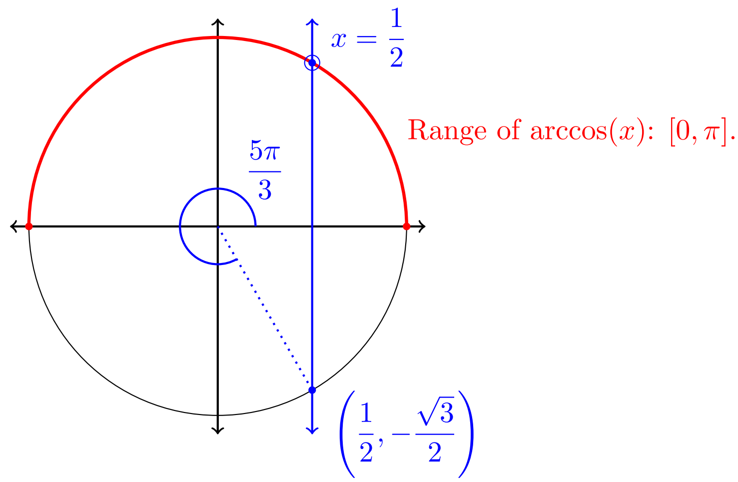


Figure 8.5: Calculating $\arccos(\cos(5\pi/3))$.

3. $\sin(\arcsin(-\sqrt{3}/2)) = -\sqrt{3}/2$, since $\sin(\arcsin(x)) = x$ for all x in the domain of $\arcsin(x)$.

4. $\cos(\arccos(3/2))$ is undefined because $3/2$ is not in the domain of $\arccos(x)$.

5. $\arctan(\tan(-\pi/4)) = -\pi/4$ because $-\pi/4$ is in the range of $\arctan(x)$.

6. $\arcsin(\sin(7\pi/4)) = -\pi/4$, as shown in the figure below. Note that $7\pi/4$ looks like it lies in the range of $\arcsin(x)$, but we must convert to an angle in $[-\pi/2, \pi/2]$, and so the answer is $-\pi/4$.

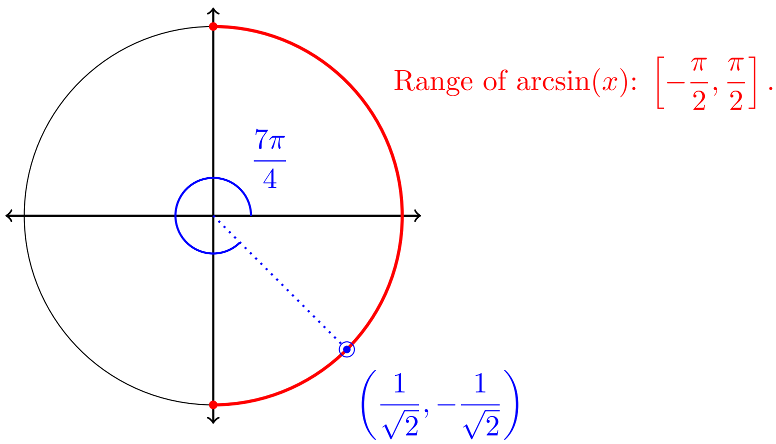


Figure 8.6: Calculating $\arcsin(\sin(7\pi/4))$.

7. $\tan(\arctan(-100)) = -100$, because $\tan(\arctan(x)) = x$ for all real numbers x.

8. $\arccos(\cos(-\pi)) = \pi$, as shown in the figure below. Note that it looks like $-\pi$ is in the range of $\arccos(x)$, but we must convert to an angle in the range of $\arccos(x)$, which is $[0, \pi]$. So the answer is π.

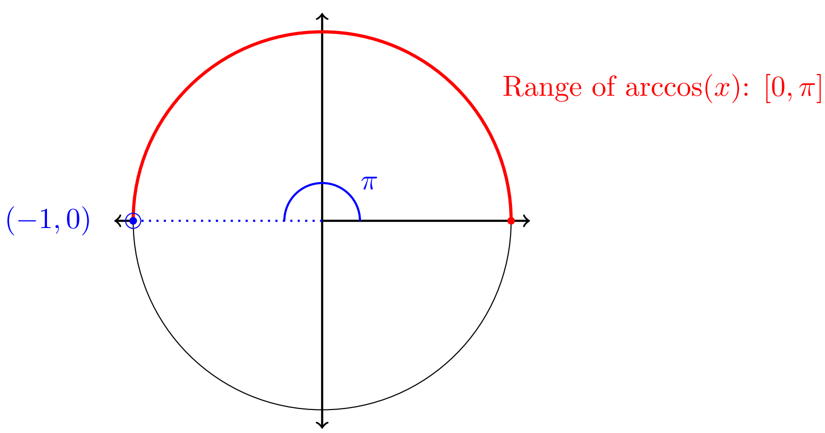


Figure 8.7: Calculating $\arccos(\cos(-\pi))$.

9. $\arctan(\tan(5\pi/4)) = \pi/4$, as shown in the figure below.

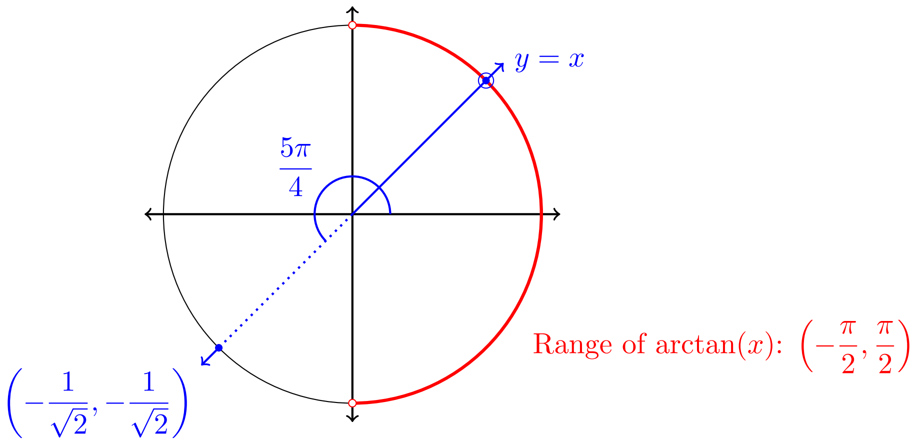


Figure 8.8: Calculating $\arctan(\tan(5\pi/4))$.

10. $\arcsin(\sin(4\pi/3)) = -\pi/3$, as shown in the figure below.

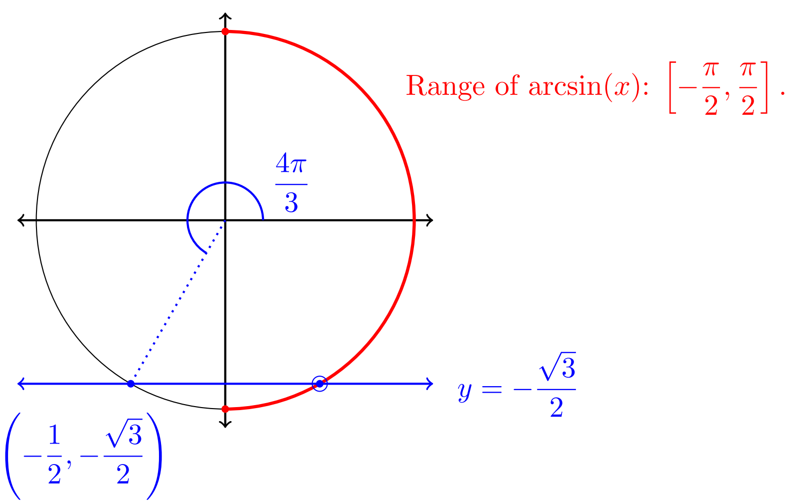


Figure 8.9: Calculating $\arcsin(\sin(4\pi/3))$.

8.3 Inverse Trigonometry II

Derivative of $\arcsin(x)$.

Now that we have a good understanding of the inverse trigonometric functions, it's time to look at their derivatives. We'll be able to make some headway using implicit differentiation. Let's start by writing $y = \arcsin(x)$ as $x = \sin(y)$, always assuming that y is in the range of $\arcsin(x)$. Now use implicit differentiation:

$$\begin{aligned} x &= \sin(y) \\ \frac{d}{dx}x &= \frac{d}{dx}\sin(y) \\ 1 &= \cos(y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\cos(y)} \end{aligned}$$

Now the question is: what do we do with $\cos(y)$? Since $\arcsin(x)$ is a function of x, our derivative should also be a function of x. Here, we use one of the Pythagorean Identities from trigonometry: for any θ, $\sin^2(\theta) + \cos^2(\theta) = 1$. We'll substitute y in for θ and solve.

$$\begin{aligned} \sin^2(y) + \cos^2(y) &= 1 \\ \cos^2(y) &= 1 - \sin^2(y) \\ \cos^2(y) &= 1 - x^2 && \text{since } x = \sin(y) \\ \cos(y) &= \sqrt{1 - x^2} \\ \frac{dy}{dx} &= \frac{1}{\cos(y)} \\ \frac{dy}{dx} &= \frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

We do have address the fact that we solved for $\cos(y)$ as $+\sqrt{1 - x^2}$ instead of $-\sqrt{1 - x^2}$. Recall that the graph of $\arcsin(x)$ is always increasing, which means its derivative always has to be positive. That's why we could take the positive square root.

Derivative of $\arccos(x)$.

We can use the same approach as we did for $\arcsin(x)$. But there is an easier way if we look at the right triangle in Figure 8.10.

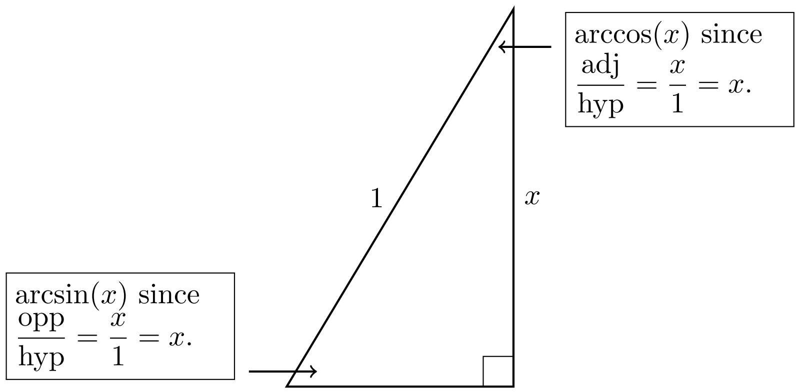


Figure 8.10: Showing $\arcsin(x) + \arccos(x) = \dfrac{\pi}{2}$.

The angle at the upper right is $\arccos(x)$ because the cosine is $\dfrac{\text{adj}}{\text{hyp}}$, and the adjacent side relative to this angle is x and the hypotenuse is 1. Similarly, the angle at the bottom left is $\arcsin(x)$ because the sine is $\dfrac{\text{opp}}{\text{hyp}}$, and the opposite side relative to this angle is x and the hypotenuse is 1.

Since the angles of a triangle add up to 180°, and since there's already a right angle, which is 90°, the other two angles must add up to 90°. But in Calculus, we always use radians, and so

$$\arcsin(x) + \arccos(x) = \frac{\pi}{2}.$$

This helps because we can solve for $\arccos(x)$ and then use what we just learned.

$$\begin{aligned}
\arcsin(x) + \arccos(x) &= \frac{\pi}{2} \\
\arccos(x) &= \frac{\pi}{2} - \arcsin(x) \\
\frac{d}{dx}\arccos(x) &= \frac{d}{dx}\frac{\pi}{2} - \frac{d}{dx}\arcsin(x) \\
&= 0 - \frac{1}{\sqrt{1-x^2}} \\
&= -\frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

Derivative of $\arctan(x)$.

While we know that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, we do not have a similar formula for $\arctan(x)$. So we have to use implicit differentiation again. We write $x = \tan(y)$, again assuming that y is in the range of $\arctan(x)$.

$$\begin{aligned}
x &= \tan(y) \\
\frac{d}{dx}x &= \frac{d}{dx}\tan(y) \\
1 &= \sec^2(y)\frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{1}{\sec^2(y)}
\end{aligned}$$

Again, we need to change the y to x. We can start with the identity we used before, and divide by $\cos^2(\theta)$.

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 \\
\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} &= \frac{1}{\cos^2(\theta)} \\
\tan^2(\theta) + 1 &= \sec^2(\theta)
\end{aligned}$$

So let's substitute this back in and simplify, using y instead of θ.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\sec^2(y)} \\
&= \frac{1}{\tan^2(y) + 1} \\
&= \frac{1}{x^2 + 1} && \text{since } x = \tan(y)
\end{aligned}$$

Examples

1. Find $h'(x)$ if $h(x) = x\arctan(x)$.

   Here, we need the product rule with $f(x) = x$ and $g(x) = \arctan(x)$.

$$\begin{aligned} h(x) &= x\arctan(x) \\ h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= x \cdot \frac{1}{x^2+1} + \arctan(x)\cdot 1 \\ &= \frac{x}{x^2+1} + \arctan(x). \end{aligned}$$

2. Find $h'(x)$ if $h(x) = \arccos(x^2)$.

   Use the Chain Rule, with $f(x) = \arccos(x)$ and $g(x) = x^2$.

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= -\frac{1}{\sqrt{1-(g(x))^2}} \cdot 2x \\ &= -\frac{2x}{\sqrt{1-(x^2)^2}} \\ &= -\frac{2x}{\sqrt{1-x^4}} \end{aligned}$$

3. Find $h'(x)$ if $h(x) = \arcsin(2x-1)$. Simplify.

   Use the Chain Rule, with $f(x) = \arcsin(x)$ and $g(x) = 2x-1$.

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= \frac{1}{\sqrt{1-(g(x))^2}} \cdot 2 \\ &= \frac{2}{\sqrt{1-(2x-1)^2}} \\ &= \frac{2}{\sqrt{1-(4x^2-4x+1)}} \\ &= \frac{2}{\sqrt{4x-4x^2}} \\ &= \frac{2}{\sqrt{4}\sqrt{x-x^2}} \\ &= \frac{1}{\sqrt{x-x^2}} \end{aligned}$$

Homework

1. If $h(x) = x^2 \arcsin(x)$, find $h'(x)$.
2. If $h(x) = \arctan(2x + 1)$, find $h'(x)$.
3. If $h(x) = \arccos(1 - x)$, find $h'(x)$.
4. If $\arcsin(y) + y = x$, find $\dfrac{dy}{dx}$.
5. Note that $\lim\limits_{x \to \infty} \dfrac{1}{x^2 + 1} = 0$. How does this relate to the horizontal asymptotes of the graph of $y = \arctan(x)$?
6. Find an equation of the tangent line to $\arccos(x)$ at $x = \dfrac{\sqrt{3}}{2}$. Check with desmos. If you type "y=arccos(x)" into desmos, you'll get the graph of $y = \arccos(x)$.

Solutions

1. Use the Product Rule, with $f(x) = x^2$ and $g(x) = \arcsin(x)$.

$$\begin{aligned}
h(x) &= f(x)g(x) \\
h'(x) &= f(x)g'(x) + g(x)f'(x) \\
&= x^2 \cdot \frac{1}{\sqrt{1-x^2}} + \arcsin(x) \cdot 2x \\
&= \frac{x^2}{\sqrt{1-x^2}} + 2x\arcsin(x)
\end{aligned}$$

2. Use the Chain Rule, with $f(x) = \arctan(x)$ and $g(x) = 2x + 1$.

$$\begin{aligned}
h(x) &= f(g(x)) \\
h'(x) &= f'(g(x))g'(x) \\
&= \frac{1}{(g(x))^2+1} \cdot 2 \\
&= \frac{2}{(2x+1)^2+1} \\
&= \frac{2}{(4x^2+4x+1)+1} \\
&= \frac{2}{4x^2+4x+2} \\
&= \frac{2}{2(2x^2+2x+1)} \\
&= \frac{1}{2x^2+2x+1}
\end{aligned}$$

3. Use the Chain Rule, with $f(x) = \arccos(x)$ and $g(x) = 1 - x$.

$$\begin{aligned}
h(x) &= f(g(x)) \\
h'(x) &= f'(g(x))g'(x) \\
&= -\frac{1}{\sqrt{1-(g(x))^2}} \cdot (-1) \\
&= \frac{1}{\sqrt{1-(1-x)^2}} \\
&= \frac{1}{\sqrt{1-(1-2x+x^2)}} \\
&= \frac{1}{\sqrt{2x-x^2}}
\end{aligned}$$

4. For this problem, we need to use implicit differentiation. After Step 3, we multiply through by $\sqrt{1-y^2}$ to eliminate fractions to make the algebra easier.

$$\begin{aligned}
\arcsin(y) + y &= x \\
\frac{d}{dx}\arcsin(y) + \frac{d}{dx}y &= \frac{d}{dx}x \\
\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} + \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} + \sqrt{1-y^2}\frac{dy}{dx} &= \sqrt{1-y^2} \\
(1+\sqrt{1-y^2})\frac{dy}{dx} &= \sqrt{1-y^2} \\
\frac{dy}{dx} &= \frac{\sqrt{1-y^2}}{1+\sqrt{1-y^2}}
\end{aligned}$$

5. We know that $y = \arctan(x)$ has a horizontal asymptote at $y = \dfrac{\pi}{2}$. This means that as $x \to \infty$, the curve has to flatten out in order to approach the asymptote without crossing it. So as the curve flattens out, the slope of the tangent line gets closer and closer to 0.

6. First, we find the slope of the tangent line by plugging in $\dfrac{\sqrt{3}}{2}$ into the derivative.

$$\begin{aligned}
m &= -\frac{1}{\sqrt{1-(\sqrt{3}/2)^2}} \\
&= -\frac{1}{\sqrt{1-3/4}} \\
&= -\frac{1}{\sqrt{1/4}} \\
&= -\frac{1}{1/2} \\
&= -2
\end{aligned}$$

Since $\arccos(\sqrt{3}/2) = \pi/6$, we use the point $(\sqrt{3}/2, \pi/6)$.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - \pi/6 &= -2(x - \sqrt{3}/2) \\
y - \pi/6 &= -2x + \sqrt{3} \\
y &= -2x + \sqrt{3} + \pi/6
\end{aligned}$$

8.4 Summary of Continuity and Differentiation

As we did with limits, we'll now summarize where continuity and differentiation are important in calculus.

Continuity

1. **Graphs.** Continuity is helpful in decribing features of graphs. If a is in the domain of a function $f(x)$, we say that

   (a) $f(x)$ has a removable discontinuity at a if both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal, but $f(a)$ is *not* equal to this value.

   (b) $f(x)$ has an essential discontinuity at a if both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, but are not equal to each other.

   (c) $f(x)$ is continuous at a if both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal, and $f(a)$ *is* equal to this value.

   (d) The function $f(x)$ is continuous if it is continuous at all points a in the domain.

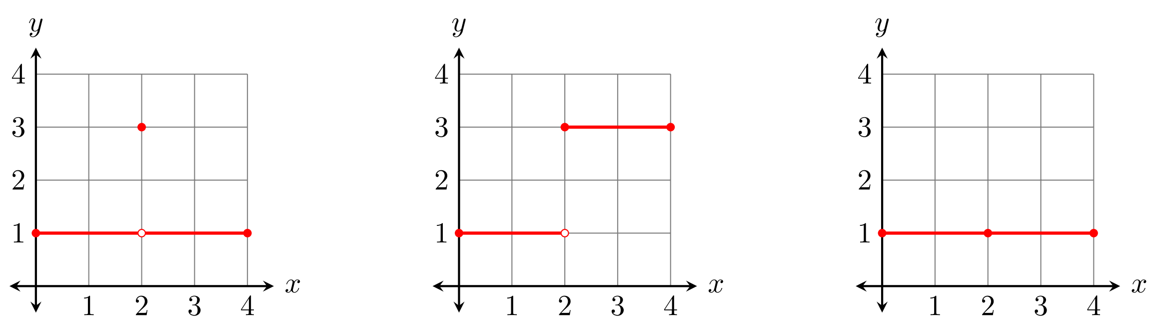


Figure 8.11: At $x = 2$: Removable discontinuity (left), essential discontinuity (middle), continuous (right).

See Section 6.1.

2. **Intermediate Value Theorem (IVT).** The Intermediate Value Theorem is usually stated as follows.

> Suppose $f(x)$ is a continuous function defined on a closed interval $[a, b]$. If $f(a) \neq f(b)$, and if c is between $f(a)$ and $f(b)$, then there is some x_0 in the open interval (a, b) such that $f(x_0) = c$.

We applied this by showing that two curves must intersect. The geometry is this: if the blue curve $f(x)$ is above the red curve $g(x)$ at one endpoint of a closed interval, and the red curve is above the blue curve at the other endpoint, they have to cross somewhere in middle, assuming the curves are continuous.

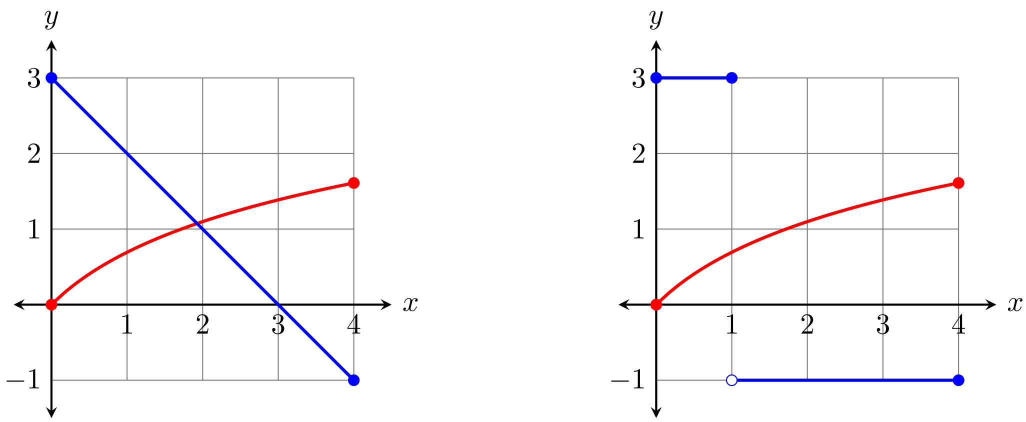


Figure 8.12: Continuous curves (left), at least one curve not continuous (right).

On the right in Figure 8.12, you can see that if the blue curve is above the red curve on the left, but below on the right, and one of the curves is *not* continuous, the curves do not have to intersect. But if both curves are continuous, they *must* intersect.

How do we use the IVT to show this? Given the geometry of the curves, the function $f(x) - g(x)$ must be negative at one endpoint and positive at the other. Since 0 lies between any negative and positive number, there is a point x_0 in the interval where $f(x_0) - g(x_0) = 0$, which means $f(x_0) = g(x_0)$. Therefore the curves intersect at x_0.

Details may be found in Section 6.4.

3. **Extreme Value Theorem (EVT).** The Extreme Value Theorem states:

> If a function is defined on a closed interval and is continuous, both a global minimum and a global maximum exist.

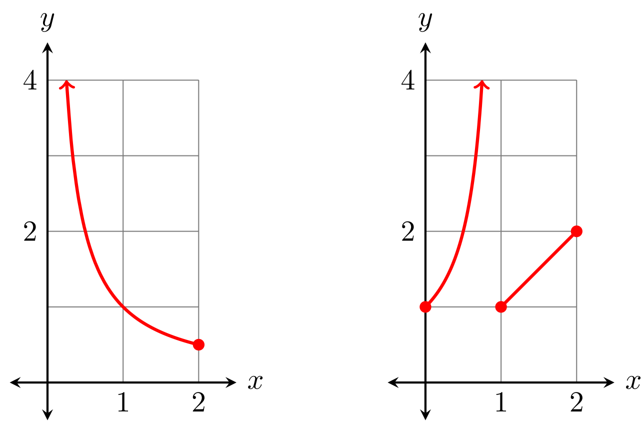


Figure 8.13: The importance of a closed interval (left), and continuiuty (right).

It is important that the interval is closed, since on the left of Figure 8.13, if the domain is $(0, 2]$, you can have a vertical asymptote. And if the function is not continuous, you might have a vertical asymptote inside the interval, as shown on the right of Figure 8.13.

How do you find global extrema? You have to look at values of the function where $f'(x) = 0$ or where the derivative is undefined, and also at the endpoints. So this theorem involves both continuity and differentiation.

See Sections 6.2 and 6.3.

Differentiation

4. **Finding rates of change.** One important use of the derivative in all sciences is to find rates of change. We looked at a few primary examples. If you had a function of displacement in km as a function of time in hours, you would find the rate of change – which is just the velocity – in units of km/hr. You find the rate of change by taking the derivative of the function.

   Another example we looked is exponential growth of organisms. Bacteria in a Petri dish start to grow exponentially, but as the dish gets full, the growth rate slows down. But an exponential function is a good model for what happens at the beginning.

   The variable P (for population) is often used to describe exponential growth. When you take $P'(t)$, where t is in hours, you are finding a rate of change. That is, you're looking at approximately how many bacteria per hour the population is growing. Again, you do this by taking the derivative and then plugging in your given value of t.

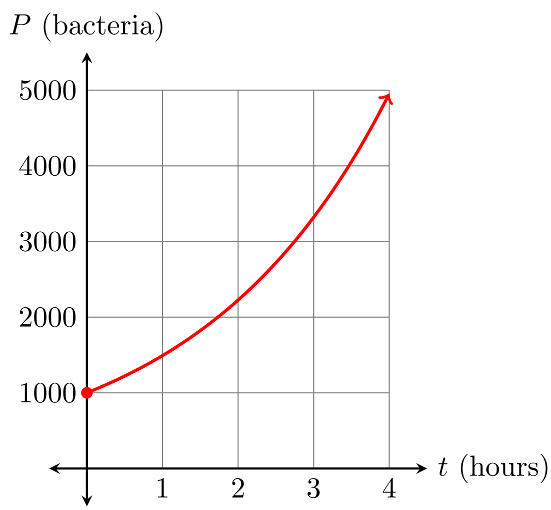
   

   Figure 8.14: Graph of $P(t) = 1000e^{0.4t}$.

   More about exponential functions may be found in Section 5.1.

5. **Finding equations of tangent lines.** The rate of change is *also* the slope of a tangent line to the graph of a function. Given a point on a graph, we can use the derivative to find the slope of the tangent line, and then find an equation for the tangent line.

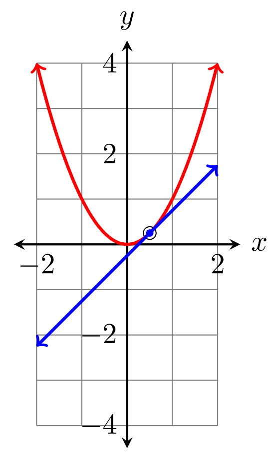
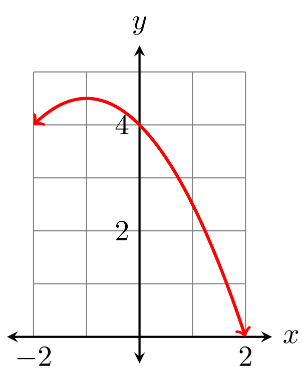


Figure 8.15: Tangent line on a graph.

See Section 2.3 for examples of finding equations of tangent lines.

6. **Find where a function is Increasing/decreasing.** The derivative is also useful to find out where a function is increasing or decreasing. When $f'(x) > 0$, the function is increasing, and when $f'(x) < 0$, the function is decreasing (see Figure 8.16 (left)). When $f'(x) = 0$, more work has to be done. In this case, there could be a local extremum, or the function could also be decreasing or increasing (as in Figure 8.16 (right)).



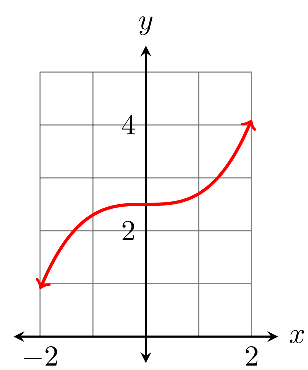


Figure 8.16: Increasing on $(-2, -1)$ and decreasing on $(-1, 2)$ (left), and increasing when $f'(0) = 0$ (right).

See Section 2.3 for a discussion of these ideas.

7. **Determining concavity and finding points of inflection.** Here, we need the second derivative. When $f''(x) > 0$, the graph is concave up (as in the left of Figure 8.17). When $f''(x) < 0$, the graph is concave down (see the middle of Figure 8.17). We do check when $f''(x) = 0$ to find inflection points, but more work is needed because there can also be a local minimum or maximum when $f''(x) = 0$ (as in the right of Figure 8.17). In this case, we either need a graph, or if we don't have one, making a sign chart is necessary.

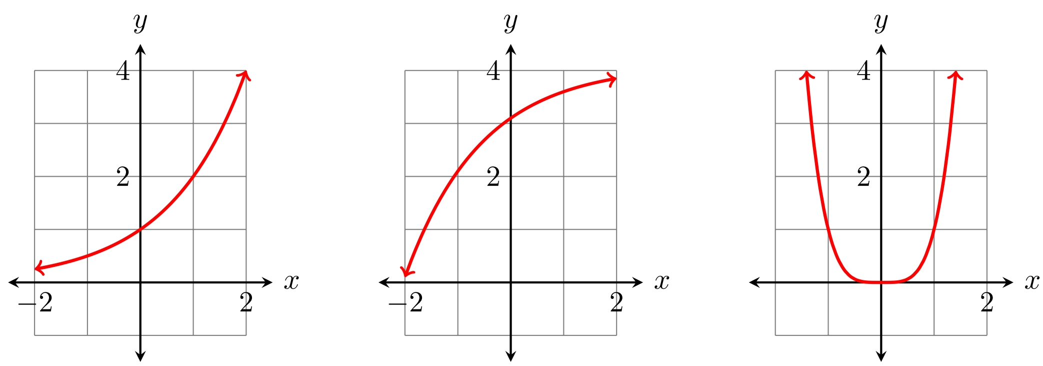


Figure 8.17: Graph of $f''(x) > 0$ (left), $f''(x) < 0$ (middle), and $f''(0) = 0$ (right).

See Section 4.2.

8. **Finding local and global extrema.** We find local extrema by solving $f'(x) = 0$ or seeing where $f'(x)$ does not exist. When $f'(x) = 0$, we can use the second derivative to see if the extremum is a minimum of maximum. If $f''(x) > 0$, the graph is concave up, and so it is a local minimum (see the left of Figure 8.18). If $f''(x) < 0$, the graph is concave down, and so it is a local maximum (middle of Figure 8.18). When $f''(x) = 0$, we need to make a sign chart, since it is possible there could be a local minimum (as in the right of Figure 8.18), a local maximum, or an inflection point.

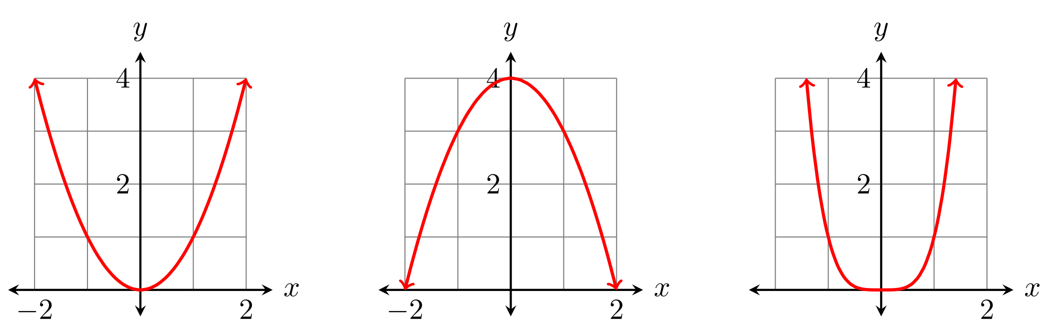


Figure 8.18: Graph of $f''(0) > 0$ (left), $f''(0) < 0$ (middle), and $f''(0) = 0$ (right).

See Section 6.2 for more discussion about local extrema.

See the Continuity section for a discussion of global extrema.

9. **L'Hôpital's Rule.** Differentiation is needed to use L'Hôpital's Rule, which is used when limits are of the form $\frac{\pm\infty}{\pm\infty}$, $\frac{0}{0}$, or $\pm\infty \cdot 0$. Suppose that $f(x)$ and $g(x)$ are functions, and a is either a real number or $\pm\infty$. Then

   (a) If $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$) as $x \to a$, L'Hôpital's Rule says that

   $$\lim_{x\to a} \frac{f(x)}{g(x)} \stackrel{\text{LR}}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

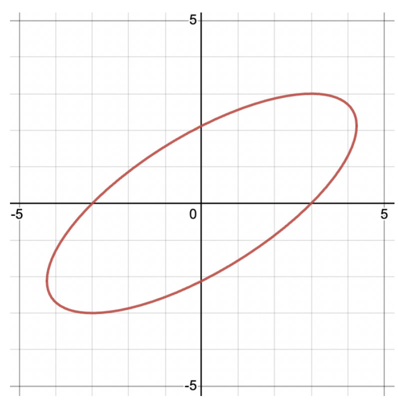
   (b) If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, L'Hôpital's Rule says that

   $$\lim_{x\to a} \frac{f(x)}{g(x)} \stackrel{\text{LR}}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

   (c) If one of $f(x)$ and $g(x)$ goes to 0 and the other goes to $\pm\infty$ as $x \to a$, you must rewrite by moving one of the function to the denominator and then applying L'Hôpital's Rule.

   See Section 7.2 for a discussion of (a) and (b), and see Section 7.3 for a discussion of (c).

10. **Finding tangents to general curves.** Many curves – like circles, ellipses, and hyperbolas are *not* graphs of functions because they fail the vertical line test. One such example is the ellipse $x^2 - 2xy + 2y^2 = 9$, shown below.



In this case, we do not have a function $y = f(x)$, but rather we say that y is defined *implicitly*. In such cases, we can use implicit differentiation to find $\frac{dy}{dx}$. Once we find $\frac{dy}{dx}$, we can use the derivative to find tangent lines, asymptotes, etc.

See Section 8.1 for examples of how implicit differentiation is used.

Finding Derivatives

Here is a summary of all the derivatives we know (that is, you can just use them at any time without justification), and the basic rules of differentiation.

1. $\frac{d}{dx}\sin(x) = \cos(x)$.
2. $\frac{d}{dx}\cos(x) = -\sin(x)$.
3. $\frac{d}{dx}\tan(x) = \sec^2(x)$.
4. $\frac{d}{dx}e^x = e^x$.
5. $\frac{d}{dx}\ln x = \frac{1}{x}$.
6. $\frac{d}{dx}b^x = b^x \ln b$.
7. $\frac{d}{dx}\log_b(x) = \frac{1}{x \ln b}$.
8. $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$.
9. $\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$.
10. $\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1}$.
11. The Power Rule: When $n > 0$,
$$\frac{d}{dx}x^n = nx^{n-1}.$$
12. The Sum Rule:
$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x).$$
13. The Difference Rule:
$$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x).$$
14. The Constant Multiple Rule:
$$\frac{d}{dx}(cf(x)) = cf'(x).$$
15. The Product Rule:
$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + g(x)f'(x).$$
16. The Quotient Rule:
$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$
17. The Chain Rule:
$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

8.5 Calculus and Graphing

Using calculus to study graphs has been a common theme throughout the course. We've looked at different aspects of graphs at different times. Here, we look at some examples and apply everything we've learned. We'll look at polynomials and rational functions. For each graph, we will:

1. Determine horizontal and vertical asymptotes, if any;
2. Determine local minima and maxima, if any;
3. Determine intervals where the function is increasing and decreasing;
4. Determine inflection points, if any;
5. Determine intervals on which the graph is concave up or concave down.

We will not focus on the algebra of derivatives of rational functions – this can get hairy. So we'll use software to calculate the derivatives for us. Also, we won't discuss how to determine the information if you don't have a graph. The idea is that you *do* have a graph – and any time you'd need a graph of a function in real life, you would use software. So the emphasis here is on describing various features of a graph using Calculus.

We will be working with desmos.com to aid in our explorations.

Example 1

Our first example is the polynomial $f(x) = 3x^5 - 5x^3$, which is ◯1. The first two derivatives are:

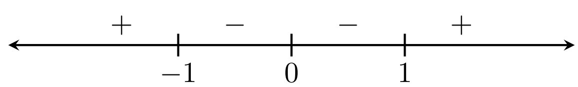
$$f'(x) = 15x^4 - 15x^2, \quad f''(x) = 60x^3 - 30x.$$

1. Asymptotes. We know that polynomials *never* have any asymptotes. Their behavior as $x \to -\infty$ and $x \to \infty$ is determined by the highest degree term.
2. Local Minima and Maxima. We need to solve $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ 15x^4 - 15x^2 &= 0 \\ 15x^2(x^2 - 1) &= 0 \\ 15x^2(x+1)(x-1) &= 0 \\ x &= -1, 0, 1 \end{aligned}$$

Since $f''(-1) = -30 < 0$, the graph is concave down at $x = -1$, so there is a local maximum at $(-1, 2)$. Since $f''(1) = 30 > 0$, the graph is concave up at $x = 1$, so there is a local minimum at $(1, 2)$. But at $x = 0$, we have $f''(0) = 0$, so we have to make a sign chart. This really isn't more work, since we need a sign chart to find out where the function is increasing and decreasing. We take test points of -2, $-\frac{1}{2}$, $\frac{1}{2}$, and 2. Evaluating:

$$f'(-2) = 180, \quad f'\left(-\frac{1}{2}\right) = -\frac{45}{16}, \quad f'\left(\frac{1}{2}\right) = -\frac{45}{16}, \quad f'(2) = 180.$$



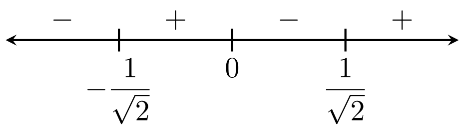
So, since we are decreasing on both sides of $x = 0$, there is an inflection point at $(0, 0)$. All this can be visually verified by looking at the graph.

3. Intervals of Increase and Decrease. We've already created a sign chart for $f'(x)$, so we can read off the intervals where the graph is increasing and decreasing. The function is increasing on $(-\infty, -1)$ and $(1, \infty)$ since $f'(x) > 0$ there, and decreasing on $(-1, 1)$ since $f'(x) < 0$ there.
4. Inflection Points. To find possible inflection points, we solve $f''(x) = 0$.

$$\begin{aligned} f''(x) &= 0 \\ 60x^3 - 30x &= 0 \\ 30x(2x^2 - 1) &= 0 \\ x &= 0 \\ 2x^2 - 1 &= 0 \\ 2x^2 &= 1 \\ x^2 &= \frac{1}{2} \\ x &= -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \end{aligned}$$

We know that there is an inflection point at $(0,0)$ from previous work. Since $\frac{1}{\sqrt{2}} \approx 0.7$, we can make a sign chart with test points -1, $-\frac{1}{2}$, $\frac{1}{2}$, and 1. Evaluating:

$$f''(-1) = -30, \quad f''\left(-\frac{1}{2}\right) = \frac{15}{2}, \quad f''\left(\frac{1}{2}\right) = -\frac{15}{2}, \quad f''(1) = 30.$$



Since the concavity changes at $-1/\sqrt{2}$ and $1/\sqrt{2}$, then there are inflection points at the points $(-1/\sqrt{2}, 7/4\sqrt{2})$ and $(1/\sqrt{2}, -7/4\sqrt{2})$. Since we want to visually inspect the graph, you can use a calculator to approximate these points as $(-0.7, 1.2)$ and $(0.7, -1.2)$. Or select ◯2, which is the equation $x = \frac{1}{\sqrt{2}}$. If you zoom in on this point, you should be able to see that the concavity changes there.

5. Intervals of Concavity. Since we have a sign chart for $f''(x)$, we see that the graph is concave down on $(-\infty, -1/\sqrt{2})$ and $(0, 1/\sqrt{2})$ (since $f''(x) < 0$ there), and the graph is concave up on $(-1/\sqrt{2}, 0)$ and $(1/\sqrt{2}, \infty)$ (since $f''(x) > 0$ there).

Assessment Expectations: For a problem like this, you would be given a graph and both the derivatives. You would only be asked to do one or two out of the five parts of the problem. You will need to make a sign chart for at least one of the parts.

Example 2

We now tackle the rational function $f(x) = \dfrac{x}{x^2+1}$, which is ◯3. The first two derivatives are:

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}, \quad f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

1. Asymptotes. The degree of the numerator is $N = 1$ and the degree of the denominator is $D = 2$. Since $N < D$, we know that $y = 0$ is a horizontal asymptote.

   Note that the denominator is *always* positive. Since it can never be 0, there are no vertical asymptotes.

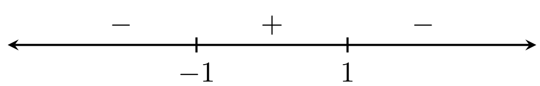
2. Local Minima and Maxima. We need to solve $f'(x) = 0$, so we set the numerator of $f'(x)$ equal to 0.

$$\begin{aligned} 1 - x^2 &= 0 \\ (1+x)(1-x) &= 0 \\ x &= -1, 1 \end{aligned}$$

   Since $f''(-1) = 1/2 > 0$, the graph is concave up at $x = -1$, so there is a local minimum at $(-1, -1/2)$. Since $f''(1) = -1/2 < 0$, the graph is concave down at $x = 1$, so there is a local maximum there at $(1, 1/2)$. These points are easily visible on the graph.

3. Intervals of Increase and Decrease. Since we didn't need a sign chart for the local extrema, we make one now. Easy test points are -2, 0, and 2. Evaluating:

$$f'(-2) = -\frac{3}{25}, \quad f'(0) = 1, \quad f'(2) = -\frac{3}{25}.$$



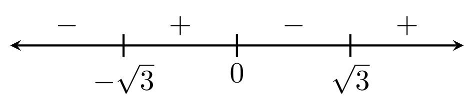
   Now we can read off the intervals where the graph is increasing and decreasing. The function is increasing on $(-1, 1)$ since $f'(x) > 0$ there, and decreasing on $(-\infty, -1)$ and $(1, \infty)$ since $f'(x) < 0$ there.

4. Inflection Points. To find possible inflection points, we set the numerator of $f''(x)$ equal to 0.

$$\begin{aligned} 2x(x^2-3) &= 0 \\ x &= 0 \\ x^2 - 3 &= 0 \\ x^2 &= 3 \\ x &= -\sqrt{3}, \sqrt{3} \end{aligned}$$

Since $\sqrt{3} \approx 1.7$, we can choose test points -2, -1, 1, and 2. Evaluating:

$$f''(-2) = -\frac{4}{125}, \quad f''(-1) = \frac{1}{2}, \quad f''(1) = -\frac{1}{2}, \quad f''(2) = \frac{4}{125}.$$



Since the concavity changes at $-\sqrt{3}$, 0, and $\sqrt{3}$, then there are inflection points at the points $(-\sqrt{3}, -\sqrt{3}/4)$, $(0,0)$, and $(\sqrt{3}, \sqrt{3}/4)$. Since we want to visually inspect the graph, you can use a calculator to approximate two of these points as $(-1.7, -0.4)$ and $(1.7, 0.4)$. Or select ◯4, which is the equation $x = \sqrt{3}$. If you zoom in on this point, you should be able to see that the concavity changes there.

5. Intervals of Concavity. Since we already have a sign chart for $f''(x)$, we see that the graph is concave down on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$ (since $f''(x) < 0$ there), and the graph is concave up on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$ (since $f''(x) > 0$ there).

Example 3

We now explore the rational function $f(x) = \dfrac{x}{x^2 - 1}$, which is ◯5. The first two derivatives are:

$$f'(x) = -\frac{x^2+1}{(x^2-1)^2}, \quad f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3}.$$

1. Asymptotes. The degree of the numerator is $N = 1$ and the degree of the denominator is $D = 2$. Since $N < D$, we know that $y = 0$ is a horizontal asymptote.

   For vertical asymptotes, we set the denominator equal to 0.

   $$\begin{aligned} x^2 - 1 &= 0 \\ (x+1)(x-1) &= 0 \\ x &= -1, 1 \end{aligned}$$

   So there are vertical asymptotes at $x = -1$ and $x = 1$. Looking at the graph, we describe the behavior of the function at these asymptotes using limit notation:

   $$\lim_{x \to -1^-} f(x) \text{ DNE } (-\infty), \quad \lim_{x \to -1^+} f(x) \text{ DNE } (+\infty),$$

   $$\lim_{x \to 1^-} f(x) \text{ DNE } (-\infty), \quad \lim_{x \to 1^+} f(x) \text{ DNE } (+\infty).$$

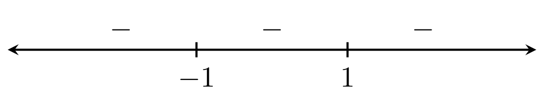
2. Local Minima and Maxima. We need to solve $f'(x) = 0$, so we set the numerator of $f'(x)$ equal to 0.

   $$\begin{aligned} x^2 + 1 &= 0 \\ x^2 &= -1 \end{aligned}$$

   Since x^2 can never be negative, we see that there are no local extrema.

3. Intervals of Increase and Decrease. To make a sign chart, we need to use the points where $f'(x) = 0$ or where the function is undefined. Note that in the previous examples, we didn't have to consider where the function was undefined since both were defined for *all* real numbers. The only places where a rational function is undefined is where there are vertical asymptotes, in this case, -1 and 1. Easy test points are -2, 0, and 2. Evaluating:

   $$f'(-2) = -\frac{5}{9}, \quad f'(0) = -1, \quad f'(2) = -\frac{5}{9}.$$



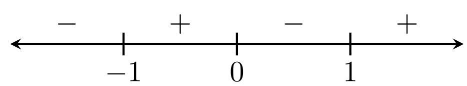
   Since $f'(x)$ is negative everywhere it's defined, then the function is decreasing on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. It is important to note that you can't just say the function is decreasing on $(-\infty, \infty)$, since all real numbers includes -1 and 1, but the function is not defined at these points. So you have to write as three separate intervals.

4. Inflection Points. To find possible inflection points, we set the numerator of $f''(x)$ equal to 0.

$$\begin{aligned} 2x(x^2+3) &= 0 \\ x &= 0 \\ x^2+3 &= 0 \\ x^2 &= -3 && \text{impossible} \end{aligned}$$

So the only possible inflection point is at $x = 0$. We make a sign chart, but again we must include -1 and 1, since we have seen examples where the concavity changes as you hop over a vertical asymptote. Easy test points are -2, $-\frac{1}{2}$, $\frac{1}{2}$, and 2. Evaluating:

$$f''(-2) = -\frac{28}{27}, \quad f''\left(-\frac{1}{2}\right) = \frac{208}{27}, \quad f''\left(\frac{1}{2}\right) = -\frac{208}{27}, \quad f''(2) = \frac{28}{27}.$$



Since the concavity changes at 0, then there is an inflection points at the point $(0,0)$. Concavity does change at $x = -1$ and $x = 1$, but the change is *over an asymptote.* Because $f(x)$ is not defined when $x = -1$ and $x = 1$, these cannot correspond to inflection points.

5. Intervals of Concavity. Since we already have a sign chart for $f''(x)$, we see that the graph is concave down on $(-\infty, -1)$ and $(0,1)$ (since $f''(x) < 0$ there), and the graph is concave up on $(-1,0)$ and $(1,\infty)$ (since $f''(x) > 0$ there).

Homework

Analyze the following functions as done above. The first one is fairly easy, so you can get a feel for working through the steps with minimal algebra. The second one is a little more involved. When working through these, make sure to include where the vertical asymptotes are when you make your sign charts!

1. $f(x) = \dfrac{1}{x}$
2. $f(x) = \dfrac{x^2}{x^2 - 4}$

   To help you out, the first two derivatives are

$$f'(x) = -\frac{8x}{(x^2-4)^2}, \quad f''(x) = \frac{8(3x^2+4)}{(x^2-4)^3}.$$

Solutions

Problem 1

We are given that $f(x) = \dfrac{1}{x}$, which is ◯6. Using the Power Rule, the first two derivatives are:

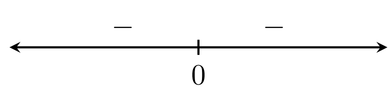
$$f'(x) = -x^{-2} = -\frac{1}{x^2}, \quad f''(x) = -(-2)x^{-3} = \frac{2}{x^3}.$$

1. Asymptotes. The degree of the numerator is $N = 0$ and the degree of the denominator is $D = 1$. Since $N < D$, we see that there is a horizontal asymptote at $y = 0$.

   For vertical asymptotes, we set the denominator equal to 0. So $x = 0$ is a vertical asymptote. Looking at the graph, we describe the behavior of the function at the asymptotes using limit notation:
   $$\lim_{x\to 0^-} f(x) \text{ DNE } (-\infty), \quad \lim_{x\to 0^+} f(x) \text{ DNE } (+\infty).$$

2. Local Minima and Maxima. We need to solve $f'(x) = 0$, so we set the numerator of $f'(x)$ equal to 0. But the numerator is 1, and so can never be 0. So there are no local minima and maxima.

3. Intervals of Increase and Decrease. To make a sign chart, we need to use the points where $f'(x) = 0$ or where the function is undefined. Thus, since $f'(x)$ is never 0, we just use $x = 0$, the vertical asymptote. Easy test points are $x = -1$ and $x = 1$. Evaluating:
   $$f'(-1) = -1, \quad f'(1) = -1.$$

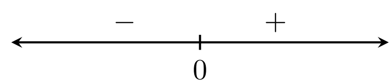
   

   Reading off the sign chart, we see that $f(x)$ is decreasing when on $(-\infty, 0)$ and $(0, \infty)$, since $f'(x) < 0$ there. Note that we cannot write this as $(-\infty, \infty)$, since this interval includes 0, but the function is not defined there.

4. Inflection Points. To find possible inflection points, we set the numerator of $f''(x)$ equal to 0. But the numerator of $f''(x)$ is 2, and so can never be 0. Thus, there are no inflection points.

5. Intervals of Concavity. There are no points where $f''(x) = 0$, so we use only $x = 0$, which corresponds to the vertical asymptote, to make our sign chart. Easy test values are -1 and 1. Evaluating:

$$f''(-1) = -2, \quad f''(1) = 2.$$



Thus, the graph of $f(x)$ is concave up on $(0, \infty)$ since $f''(x) > 0$ there, and concave down on $(-\infty, 0)$ since $f''(x) < 0$ there.

Problem 2

We are given that $f(x) = \dfrac{x^2}{x^2-4}$, which is ◯7. The first two derivatives are:

$$f'(x) = -\frac{8x}{(x^2-4)^2}, \quad f''(x) = \frac{8(3x^2+4)}{(x^2-4)^3}.$$

1. Asymptotes. The degree of the numerator is $N = 2$ and the degree of the denominator is $D = 2$. Since $N = D$, we take the ratio of the leading coefficients of the numerator and denominator – both 1 in this case – to get the horizontal asymptote $y = \dfrac{1}{1} = 1$.

   For vertical asymptotes, we set the denominator equal to 0.

$$\begin{aligned} x^2 - 4 &= 0 \\ (x+2)(x-2) &= 0 \\ x &= -2, 2 \end{aligned}$$

   So there are vertical asymptotes at $x = -2$ and $x = 2$. Looking at the graph, we describe the behavior of the function at these asymptotes using limit notation:

$$\lim_{x\to -2^-} f(x) \text{ DNE } (+\infty), \quad \lim_{x\to -2^+} f(x) \text{ DNE } (-\infty),$$

$$\lim_{x\to 2^-} f(x) \text{ DNE } (-\infty), \quad \lim_{x\to 2^+} f(x) \text{ DNE } (+\infty).$$

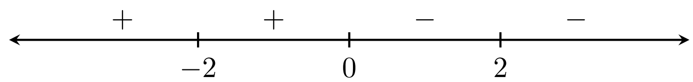
2. Local Minima and Maxima. We need to solve $f'(x) = 0$, so we set the numerator of $f'(x)$ equal to 0.

$$\begin{aligned} -8x &= 0 \\ x &= 0 \end{aligned}$$

   Since $f''(0) = -\dfrac{1}{2}$, the graph is concave down at $x = 0$. Thus, there is a local maximum at $(0, 0)$.

3. Intervals of Increase and Decrease. To make a sign chart, we need to use the points where $f'(x) = 0$ or where the function is undefined. Thus, we use $x = 0$, (just calculated), and the vertical asymptotes -2 and 2. Easy test points are -3, -1, 1, and 3. Evaluating:

$$f'(-3) = \frac{24}{25}, \quad f'(-1) = \frac{8}{9}, \quad f'(1) = -\frac{8}{9}, \quad f'(3) = -\frac{24}{25}$$



Reading off the sign chart, we see that $f(x)$ is increasing when on $(-\infty, -2)$ and $(-2, 0)$, since $f'(x) > 0$ there. Note that we cannot write this as $(-\infty, 0)$, since this interval includes -2, but the function is not defined there. The function is decreasing on $(0, 2)$ and $(2, \infty)$ since $f'(x) < 0$ there. Again, two separate intervals are needed.

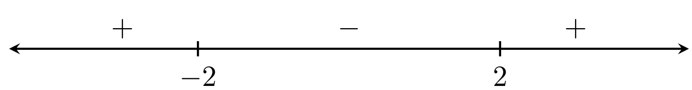
4. Inflection Points. To find possible inflection points, we set the numerator of $f''(x)$ equal to 0.

$$\begin{aligned} 8x(3^2 + 4) &= 0 \\ 3x^2 + 4 &= 0 \\ 3x^2 &= -4 \\ x^2 &= -\frac{4}{3} \qquad \text{impossible} \end{aligned}$$

Since x^2 can never be negative, there are no inflection points.

5. Intervals of Concavity. There are no points where $f''(x) = 0$, so we use only -2 and 2, which correspond to the vertical asymptotes, to make our sign chart. Easy test values are -3, 0, and 3. Evaluating:

$$f''(-3) = \frac{248}{125}, \quad f''(0) = -\frac{1}{2}, \quad f''(3) = \frac{248}{125}.$$



Thus, the graph of $f(x)$ is concave up on $(-\infty, -2)$ and $(2, \infty)$ since $f''(x) > 0$ there, and concave down on $(-2, 2)$ since $f''(x) < 0$ there.

8.6 Related Rates

We will look at another important application of implicit differentiation. Recall that in Section 8.1, we used this technique to find slopes of tangent lines to an implicitly defined curve. For example, if $y^3 + xy = x^2$, we differentiated through with respect to x and solved for $\frac{dy}{dx}$, as follows:

$$\begin{aligned} \frac{d}{dx}y^3 + \frac{d}{dx}xy &= \frac{d}{dx}x^2 \\ 3y^2\frac{dy}{dx} + x\frac{dy}{dx} + y &= 2x \\ \frac{dy}{dx} &= \frac{2x-y}{3y^2+x} \end{aligned}$$

We differentiated through with respect to x because the slope $\frac{dy}{dx}$ is the rate of change of y with respect to x.

If we differentiate an equation with with to the time t, we obtain a relationship between two or more different rates of change.

Example 1

Let's suppose you are blowing up a balloon using an air pump whose output is 3000 cm^3 of air per second (this is about 180 cubic inches). At the beginning, the balloon expands rapidly. But as it gets bigger, it expands more slowly. When the radius of the balloon is 15 cm, how fast is it expanding?

We use the term **related rates** because how fast the radius is changing clearly depends on how fast the volume is changing. The less air the pump puts out, the slower the radius will change.

To find out how these rates are related, we first need to recall the formula for the volume of a sphere in terms of its radius:

$$V = \frac{4}{3}\pi r^3.$$

Similar to what we did for implicit differentiation, we now differentiate through by t to see how the rates are related. The actual calculations are the same as before, but we're just using different variables.

$$\begin{aligned} \frac{d}{dt}V &= \frac{d}{dt}\frac{4}{3}\pi r^3 \\ \frac{dV}{dt} &= \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) \\ \frac{dV}{dt} &= 4\pi r^2\frac{dr}{dt} \end{aligned}$$

Notice there are three quantities to be substituted in the last equation: $\frac{dV}{dt}$, r, and $\frac{dr}{dt}$. We are given two of the three: $\frac{dV}{dt} = 3000$ and $r = 15$. So we substitute and use a calculator to get

$$\begin{aligned} 3000 &= 4\pi(15^2)\frac{dr}{dt} \\ \frac{dr}{dt} &= 1.06 \end{aligned}$$

Thus, when the radius of the balloon is 15 cm, the radius is growing at a rate of 1.06 cm/s. Be sure that your final answer *always* has the appropriate units.

Let's look at the equation

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

in more detail. Since $\frac{dV}{dt}$ is constant, we may plug in and solve for $\frac{dr}{dt}$.

$$\begin{aligned} 3000 &= 4\pi r^2\frac{dr}{dt} \\ \frac{dr}{dt} &= \frac{3000}{4\pi r^2} \\ &\approx \frac{238.7}{r^2} \end{aligned}$$

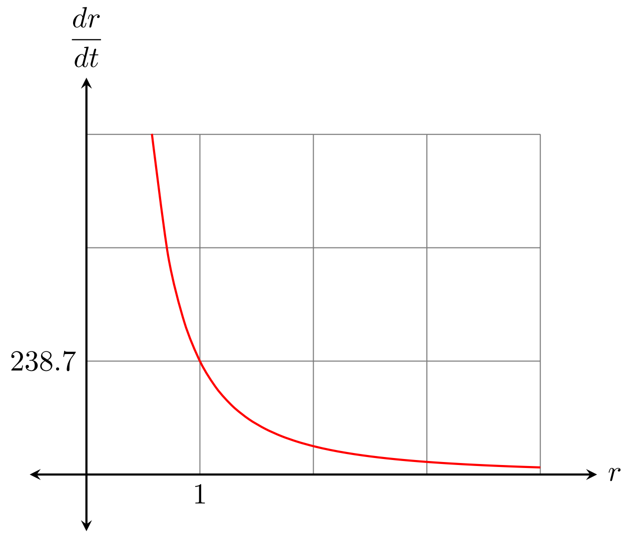


Figure 8.19: Graph of $\frac{dr}{dt} = \frac{238.7}{r^2}$

A graph of this relationship is shown in Figure 8.19. How do we interpret this graph? Since the air is pumped at a constant volume, as the balloon gets bigger, the rate at which the radius expands keeps getting smaller.

The essential point is this. Since we have an equation which relates radius and volume (the equation for the volume of a sphere), by differentiating with respect to t, we get an equation which relates the rates of change of the volume and the radius.

Example 2

Large oil spills may release 10,000 or more liters of oil per day. A commonly used figure for the thickness of an oil spill is 1 mm. If a spill leaks 10,000 liters a day at a thickness of 1 mm and extends out in a circular area from the origin of the spill, at what rate is the radius of the spill changing per hour at the end of the first day?

The basic equation we're considering is the area of a circle in terms of its radius:

$$A = \pi r^2.$$

Differentiating with respect to t, we get

$$\begin{aligned}
\frac{d}{dt}A &= \frac{d}{dt}\pi r^2 \\
\frac{dA}{dt} &= \pi\left(2r\frac{dr}{dt}\right) \\
\frac{dA}{dt} &= 2\pi r\frac{dr}{dt}
\end{aligned}$$

We are being asked to find $\frac{dr}{dt}$ when $t = 24$, since time is being measured in hours. Assuming that oil is leaked at a constant rate, this means that $\frac{dA}{dt}$ is constant. But what is it?

In the metric system, one liter is 1000 cm^3. 1 cm is 10 mm, and so 1 cm^3 = 1000 mm^3. This means that one liter is 1,000,000 mm^3. So 10,000 liters is 10^{10} mm^3.

Using 1 mm as the thickness of the spill, we have the area of the spill is increasing at

$$\frac{10^{10}\text{ mm}^3}{1\text{ mm}} = 10^{10}\text{ mm}^2$$

per day. So per hour, we have

$$\begin{aligned}
\frac{dA}{dt} &= \frac{10^{10}\text{ mm}^2}{24\text{ hr}} \\
&\approx 4.2 \times 10^8 \text{mm}^2/\text{hr}.
\end{aligned}$$

What about r? After one day, 10,000 liters of oil have spilled. So the total area of oil spilled, assuming 1 mm thickness, is 10^{10} mm^2, as we just calculated. Using the relationship $A = \pi r^2$, we get

$$\begin{aligned}
A &= \pi r^2 \\
10^{10} &= \pi r^2 \\
r &\approx 56{,}400
\end{aligned}$$

Now we can substitute back in and solve.

$$\begin{aligned} \frac{dA}{dt} &= 2\pi r \frac{dr}{dt} \\ 4.2 \times 10^8 &= 2\pi \cdot 56{,}400 \cdot \frac{dr}{dt} \\ \frac{dr}{dt} &\approx 1185 \end{aligned}$$

So the radius is expanding at 1185 mm/hr after the first day, which is about 3.9 ft/hr.

Example 3

You are scuba diving, and have enough air in your tank to breathe 1 cu ft/min of air for an hour (this is a figure commonly used for diving). The pressure in your tank is 3000 psi (standard pressure). At what rate is the air pressure changing in your tank when you start your dive? For comparison, air pressure at sea level is 14.7 psi.

For this example, we use Boyle's Law:

$$PV = k.$$

Here, P stands for air pressure, V stands for the volume, and k is a constant to be determined by the scenario. You can use whatever units you want for P and V as long as you're consistent. We'll use psi (pounds per square inch), and cu ft (cubic feet).

We're basically at sea level, so to start, we can use $P = 14.7$. Breathing 1 cu ft/min of air for an hour requires 60 cu ft of air. So

$$PV = 14.7 \times 60 = k = 882.$$

The units of k are psi cu ft/min. Let's use the equation $PV = 882$ and differentiate with respect to t to get an equation relating rates.

$$\begin{aligned} PV &= 882 \\ \frac{d}{dt}PV &= \frac{d}{dt}882 \\ &= 0 \end{aligned}$$

This is very similar to the work we did to find $\frac{d}{dx}xy$ for implicit differentiation. We used the Product Rule. Here, we use $f = P$ and $g = V$.

$$\begin{aligned} f(t) &= P & f'(t) &= \frac{dP}{dt} \\ g(t) &= V & g'(t) &= \frac{dV}{dt} \end{aligned}$$

Now substitute into the Product Rule.

$$\begin{aligned} \frac{d}{dt}PV &= 0 \\ f(t)g'(t) + g(t)f'(t) &= 0 \\ P\frac{dV}{dt} + V\frac{dP}{dt} &= 0 \end{aligned}$$

Where to proceed from here? At the start of the dive, we know that $P = 3000$, since we haven't released any air yet (that is, decreased the pressure). To find V at the start of the dive – that is, the volume of your tank – we use Boyle's Law.

$$\begin{aligned} PV &= 882 \\ 3000 \cdot V &= 882 \\ V &\approx 0.3 \end{aligned}$$

At the start of the dive, we are using 0.3/60 cu ft of air per minute (since the dive is for one hour), and so

$$\frac{dV}{dt} = 0.3/60 = 0.005.$$

Substituting these values in, we get:

$$\begin{aligned} P\frac{dV}{dt} + V\frac{dP}{dt} &= 0 \\ 3000 \times 0.005 + 0.3 \cdot \frac{dP}{dt} &= 0 \\ 0.3 \cdot \frac{dP}{dt} &= -15 \\ \frac{dP}{dt} &= -50 \end{aligned}$$

This means that the pressure in your tank is decreasing approximately 50 psi per minute at the beginning of your dive.

Bear in mind that this is an oversimplification. Air in tanks is actually a mixture of oxygen, helium, and nitrogen, and the proportions vary depending on how deep the dive is. But regardless, Boyle's Law is still very important in the science of scuba diving.

Homework

1. Suppose you are blowing up a balloon using an air pump whose output is 4000 cm^3 of air per second (this is about 240 cubic inches). When the radius of the balloon is 12 cm, how fast is it expanding?

2. If an oil spill leaks 12,000 liters a day at a thickness of 1.5 mm and extends out in a circular area from the origin of the spill, at what rate is the radius of the spill changing after 12 hours?

3. You are scuba diving, and have enough air in your tank to breathe 1.2 cu ft/min of air for an hour. The pressure in your tank is 2500 psi. At what rate is the air pressure changing in your tank when you start your dive? Air pressure at sea level is 14.7 psi.

Solutions

1. Here, we need to recall the formula for the volume of a sphere in terms of its radius:

$$V = \frac{4}{3}\pi r^3.$$

We now differentiate through by t to see how the rates are related.

$$\begin{aligned}\frac{d}{dt}V &= \frac{d}{dt}\frac{4}{3}\pi r^3 \\ \frac{dV}{dt} &= \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) \\ \frac{dV}{dt} &= 4\pi r^2\frac{dr}{dt}\end{aligned}$$

Notice there are three quantities to be substituted in the last equation: $\frac{dV}{dt}$, r, and $\frac{dr}{dt}$. We are given two of the three: $\frac{dV}{dt} = 4000$ and $r = 12$. So we substitute and use a calculator to get

$$\begin{aligned}4000 &= 4\pi(12^2)\frac{dr}{dt} \\ \frac{dr}{dt} &\approx 2.21\end{aligned}$$

Thus, when the radius of the balloon is 12 cm, the radius is growing at a rate of 2.21 cm/s.

2. The basic equation we're considering is the area of a circle in terms of its radius:

$$A = \pi r^2.$$

Differentiating with respect to t, we get

$$\begin{aligned}\frac{d}{dt}A &= \frac{d}{dt}\pi r^2 \\ \frac{dA}{dt} &= \pi\left(2r\frac{dr}{dt}\right) \\ \frac{dA}{dt} &= 2\pi r\frac{dr}{dt}\end{aligned}$$

We are being asked to find $\frac{dr}{dt}$ when $t = 12$, since time is being measured in hours. Assuming that oil is leaked at a constant rate, this means that $\frac{dA}{dt}$ is constant.

In the metric system, one liter is 1000 cm^3. 1 cm is 10 mm, and so 1 cm^3 = 1000 mm^3. This means that one liter is 1,000,000 mm^3. So 12,000 liters is 1.2×10^{10} mm^3.

Using 1.5 mm as the thickness of the spill, we have the area of the spill is increasing at

$$\frac{1.2 \times 10^{10}\ \text{mm}^3}{1.5\ \text{mm}} = 8 \times 10^9\ \text{mm}^2$$

per day. So per hour, we have

$$\begin{aligned}\frac{dA}{dt} &= \frac{8 \times 10^9 \text{ mm}^2}{24 \text{ hr}} \\ &\approx 3.3 \times 10^8 \text{mm}^2/\text{hr}.\end{aligned}$$

What about r? After one day, 12,000 liters of oil have spilled. After 12 hours, only 6000 liters have spilled. So the total area of oil spilled after 12 hours, assuming 1.5 mm thickness, is *half* of 8×10^9 mm^2, or 4×10^9 mm^2. Using the relationship $A = \pi r^2$, we get

$$\begin{aligned}A &= \pi r^2 \\ 4 \times 10^9 &= \pi r^2 \\ r &\approx 35{,}700\end{aligned}$$

Now we can substitute back in and solve.

$$\begin{aligned}\frac{dA}{dt} &= 2\pi r \frac{dr}{dt} \\ 3.3 \times 10^8 &= 2\pi \cdot 35{,}700 \cdot \frac{dr}{dt} \\ \frac{dr}{dt} &\approx 1471\end{aligned}$$

So the radius is expanding at 1471 mm/hr after the first day.

3. For this example, we use Boyle's Law:

$$PV = k.$$

For units, we'll use psi (pounds per square inch), and cu ft (cubic feet).

We're basically at sea level, so to start, we can use $P = 14.7$. Breathing 1.2 cu ft/min of air for an hour requires 72 cu ft of air. So

$$PV = 14.7 \times 72 = k = 1058.$$

The units of k are psi cu ft/min. Let's use the equation $PV = 1058$ and differentiate with respect to t to get an equation relating rates.

$$\begin{aligned}PV &= 1058 \\ \frac{d}{dt}PV &= \frac{d}{dt}1058 \\ P\frac{dV}{dt} + V\frac{dP}{dt} &= 0\end{aligned}$$

At the start of the dive, we know that $P = 2500$, since we haven't released any air yet (that is, decreased the pressure). To find V at the start of the dive – that is, the volume of your tank – we use Boyle's Law.

$$\begin{aligned}PV &= 1058 \\ 2500 \cdot V &= 1058 \\ V &\approx 0.4\end{aligned}$$

At the start of the dive, we are using 0.4/60 cu ft of air per minute (since the dive is for one hour), and so

$$\frac{dV}{dt} = 0.4/60 = 0.007.$$

Substituting these values in, we get:

$$\begin{aligned} P\frac{dV}{dt} + V\frac{dP}{dt} &= 0 \\ 2500 \times 0.007 + 0.4 \cdot \frac{dP}{dt} &= 0 \\ 0.4 \cdot \frac{dP}{dt} &= -17.5 \\ \frac{dP}{dt} &= -44 \end{aligned}$$

This means that the pressure in your tank is decreasing approximately 44 psi per minute at the beginning of your dive.

Chapter 9

Antidifferentiation

9.1 Antiderivatives

As we just saw in the YouTube video, the trajectory of water shooting out from a pump looks like an upside-down parabola. Why should this be?

The answer is "gravity." If there wasn't any gravity, then when water shot out of a pump, it would just keep traveling in a straight line, going higher and higher. This is called Newton's First Law of Motion. Or if you threw a baseball, it would never hit the ground, it would just keep going in the direction you threw it.

Basically, what's happening is this. When you throw something up, it wants to keep going up. But gravity wants to bring it back down. So you have two opposite forces – how hard you threw it, and how strong gravity is. It turns out that gravity wins.

Why is that? It's really hard to go to the moon – you have to counteract gravity. In order to leave the Earth's atmosphere, it turns out that you have to be going *at least* 25,000 miles an hour! The fastest baseball pitch has been clocked at 102 mph. Not even close.

We need a little more physics. Suppose a skydiver jumps out of an airplane. Once you jump, you start falling. Your *velocity* keeps increasing as you keep falling – you keep falling faster and faster. But your *accleration* is constant.

This is a remarkable fact, and physicists have been studying gravity for centuries. What's important for us is just the fact that this acceleration due to gravity is constant, at 9.8 m/s^2.

A word about units. If displacement is measured in meters and time is measured in seconds, then the velocity – the rate of change – is measured in m/s. This is the same as mph (miles per hour), except with a change of units. So since acceleration is the derivative (rate of change) of velocity, its units are m/s/s, or m/s^2.

So why is this important? We know that when we take the derivative of displacement, we get the velocity. And when we take the derivative of the velocity, we get acceleration. And we know that acceleration is constant. So how do we find displacement? We have to work *backwards.* Before, we knew displacement and used derivatives to get acceleration. Now, we know the acceleration and have to use **antiderivatives** to get the displacement.

What is an antiderivative? Since the derivative of $f(x) = x^2$ is $2x$, we say that x^2 is an antiderivative of $2x$. We say *an* antiderivative, because there is more than one. But if $g(x) = x^2 + 5$, then $g'(x)$ is *also* an antiderivative of $2x$. That's because when you take the derivative of a constant, you get 0. We usually say that *the* antiderivative of $2x$ is $x^2 + C$, where C can be any number.

Example 1

You are standing on the roof of a building which is 20 m tall (this is about 60 ft). You drop a marble from the roof. How long will it take to hit the ground?

One important note is that all objects, no matter how small or large, will take the same time to hit the ground – this is a fundamental principle in physics. Here, we ignore the effects of air resistance. If you dropped a feather, air resistance would slow it down. But if you dropped a marble and a bowling ball, they would take the same amount of time to reach the ground because air resistance would be negligible.

Let's create a coordinate system, as shown in Figure 9.1. As we did before, when working with displacement, we use $s(t)$ for displacement, $v(t)$ for velocity, and $a(t)$ for acceleration.

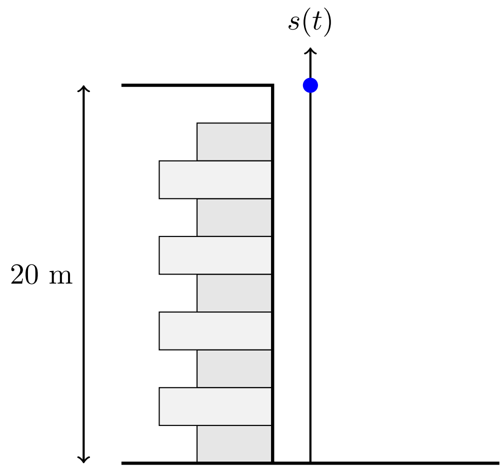


Figure 9.1: Dropping a marble from a roof.

Now let's start working backwards. We represent the fact that acceleration due to gravity is a constant 9.8 m/s^2 by

$$a(t) = -9.8.$$

Mathematically we have a negative acceleration because we are measuring displacement from the ground up. If we throw a ball *up*, gravity acts to bring it *down* – the opposite direction. The units of $a(t)$ are m/s^2.

Going backwards to find $v(t)$ – which is the antiderivative of velocity – we ask what function of t would we differentiate to get -9.8. Well, we know that the derivative of a linear function is constant, so we would guess

$$v(t) = -9.8t + C = -9.8t + v_0.$$

In physics, the term "v_0" is used instead of C, and is called the **initial velociy.** This is because

$$v(0) = -9.8(0) + v_0 = v_0.$$

The units of $v(t)$ are m/s.

What is the initial velocity in this problem? Since we are simply dropping the marble, it's just 0. Thus,

$$v(t) = -9.8t.$$

Time to go backwards again. We know that taking the derivative of t^2 will give us $2t$. So to just get t, we would have to start with $\frac{1}{2}t^2$. Now displacement is the antiderivative of velocity, and so

$$s(t) = -9.8\left(\frac{1}{2}t^2\right) + C = -4.9t^2 + s_0.$$

s_0 is called the **initial displacement** because

$$s(0) = -4.9(0^2) + s_0 = s_0.$$

In our case, we would use $s_0 = 20$, since the marble is being dropped from 20 m above ground. Thus,

$$s(t) = -4.9t^2 + 20.$$

The units of $s(t)$ are m.

Why was it so important to find $s(t)$? Our original question was to determine how long it took the marble to hit the ground. Since our coordinate system measures height above the ground, this is the same thing as asking when $s(t) = 0$, since 0 m above the ground is actually *on* the ground. So now we solve.

$$\begin{aligned}
s(t) &= 0 \\
-4.9t^2 + 20 &= 0 \\
4.9t^2 &= 20 \\
t^2 &\approx 4.08 \\
t &\approx 2.02 \text{ s}
\end{aligned}$$

Note that we took the positive square root only as the time in seconds must be a positive number. So this means that the marble will hit the ground in approximately two seconds.

The important takeaway is that because of the *physics* of falling objects, we have to start with the acceleration and work *backwards* to find the displacement. In fact, much of calculus was created in order to explain physical phenomena. This is just one more example.

Example 2

You are standing on the roof of a building which is 20 m tall (this is about 60 ft). You throw a marble down from the roof at 10 m/s (about 22 mph), not unreasonable as it is five times slower that the fastest baseball pitch. (1) How long will it take to hit the ground? (2) At what velocity does it hit the ground?

We'll work through this one a bit more quickly as we have already seen the process. As before, we start with

$$a(t) = -9.8,$$

so that

$$v(t) = -9.8t + v_0.$$

Now what is v_0? We're throwing *down* at 10 m/s, and so v_0 is -10. Remember, we're measuring displacement as the distance *up* from the ground, so anything which acts to bring our marble *down* has to be negative. Therefore,

$$v(t) = -9.8t - 10.$$

Now we work backwards once more to find $s(t)$. We use $\frac{1}{2}t^2$ as an antiderivative of t, and $-10t$ as an antiderivative of -10. Thus,

$$\begin{aligned} s(t) &= -9.8\left(\frac{1}{2}t^2\right) - 10t + s_0 \\ &= -4.9t^2 - 10t + 20, \end{aligned}$$

where we again use 20 for s_0 since our building is 20 m tall.

So to answer the first question, we must find out when $s(t) = 0$, since that corresponds to being on the ground. But to solve

$$-4.9t^2 - 10t + 20 = 0,$$

we need to remember the quadratic formula. We'll state it with the variable t since that's what we're using.

Quadratic Formula

If $at^2 + bt + c = 0$, then

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To get accuracy to one decimal, we'll write numbers to two decimal places, then round to one decimal place for our final answer. We use $a = -4.9$, $b = -10$, and $c = 20$. Be very careful with negative signs.

$$\begin{aligned} t &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(-4.9)(20)}}{2(-4.9)} \\ &= \frac{10 \pm \sqrt{100 + 392}}{-9.8} \\ &= \frac{10 \pm 22.18}{-9.8} \\ \frac{10 + 22.18}{-9.8} &\approx -3.3 \\ \frac{10 - 22.18}{-9.8} &\approx 1.2 \end{aligned}$$

As t represents a time, we choose the positive value. Thus, the marble hits the ground after about 1.2 s.

What is its velocity when it hits the ground? We substitute the value of t into the velocity equation, $v(t)$. Thus,

$$\begin{aligned} v(t) &= -9.8t - 10 \\ v(1.2) &= -9.8(1.2) - 10 \\ &\approx -21.8 \end{aligned}$$

So the marble hits the ground at about −21.8 m/s, which is about −49 mph. To make sense of your answers, it is helpful to know that to convert m/s to mph, multiply by 2.237.

Now that we've worked out the displacement, let's do it one more time, just using v_0 and s_0 without substituting in values.

$$\begin{aligned} a(t) &= -9.8 \\ v(t) &= -9.8t + v_0 \\ s(t) &= -9.8\left(\frac{1}{2}t^2\right) + v_0 \cdot t + s_0 \\ &= -4.9t^2 + v_0 t + s_0. \end{aligned}$$

There's no need to work out all the steps each time. So we summarize.

Displacement Equations

If an object is thrown with an initial velocity of v_0 m/s from a height of s_0 m, the equations for the velocity and displacement are

$$v(t) = -9.8t + v_0, \quad s(t) = -4.9t^2 + v_0 t + s_0.$$

Just a few remarks. If you throw the object up, its initial velocity will be *postive.* We used a negative initial velocity because we were throwing it *down.* Also, it is helpful to know that if you are measuring velocity in ft/s, the acceleration due to gravity is -32 ft/s^2. Units for science are almost always metric, so we'll stick to m/s in our examples. You can easily convert back on forth with online unit converters. Just google "convert meters to feet" and you'll find one.

Example 3

Suppose you throw a baseball at an angle of 60° from the horizontal at a speed of 15 m/s (which is about 34 mph). When the baseball leaves your hand, it is 2 m above the ground. (1) When will it hit the ground again? (2) How far away will the ball land? (3) By finding an equation in x and y, show that the trajectory the baseball takes is a parabola.

Note: When working out **projectile motion** problems like this in physics, angles are usually measured in degrees, not radians.

In the previous examples, we were only concerned with vertical displacement. Now we're adding in horizontal displacement as well. Because we have two displacements, we'll call the vertical displacement $y(t)$ and the horizontal displacement $x(t)$, illustrated in Figure 9.2.

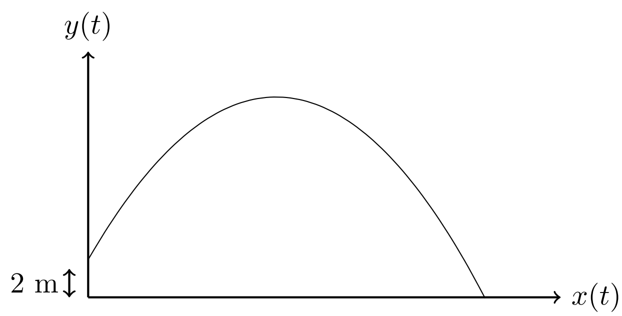


Figure 9.2: Throwing a baseball.

We know the formula for the vertical displacement from our previous work:

$$y(t) = -4.9t^2 + v_0 t + s_0.$$

What about the horizontal displacement? How does gravity affect the horizontal displacement? Not at all, actually. This is because gravity is a vertical force. A fundamental principle in physics is that a force in one direction has *no effect* on something moving in a *perpendicular* direction. Think about it: if you're running a race, does gravity slow you down? No, it doesn't. That's because your direction of motion is perpendicular to the force of gravity.

Since in our coordinate system, the x- and y-axes are perpendicular and gravity affects the y-direction, it has no effect on movement in the x-direction. In other words, the baseball moves with **constant horizontal velocity,** which we will call v_h. So when we figure out just what v_h is, then we can say that $x(t) = v_h t$. Summarizing, we have

$$y(t) = -4.9t^2 + v_0 t + s_0, \quad x(t) = v_h t.$$

We have three constants we have to figure out: v_0, s_0, and v_h. We know that $s_0 = 2$ as it is given in the problem. To figure out v_0 and v_h, we have to **decompose** the velocity into its vertical and horizontal components, as shown in Figure 9.3.

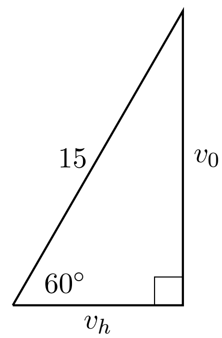


Figure 9.3: Decomposing a vector into vertical and horizontal components.

To do this, we create a right triangle as shown, and use trigonometry to find the lengths of the legs of the triangle. This decomposition of forces into components is another fundamental principle of physics. Reading off the triangle, we have

$$\sin 60^\circ = \frac{v_0}{15}, \quad \cos 60^\circ = \frac{v_h}{15}.$$

Then we get

$$v_0 = 15 \sin 60^\circ \approx 13, \quad v_h = 15 \cos 60^\circ = 7.5.$$

Thus, the vertical component of the velocity is about 13 m/s, and the horizontal component of the velocity is 7.5 m/s. So now we have equations for vertical and horizontal displacement:

$$y(t) = -4.9t^2 + 13t + 2, \quad x(t) = 7.5t.$$

We point out that we use +13 as we are throwing the ball *up*. In the previous example, we were throwing the marble *down*.

1. To see when the baseball will hit the ground, we solve $y(t) = 0$, since $y(t)$ is the vertical displacement:

$$-4.9t^2 + 13t + 2 = 0.$$

   As before, we use the quadratic formula and choose the positive solution, which is $t \approx 2.8$ s. Note that we do not need to use $x(t)$ here because we are looking at vertical displacement.

2. To see how far away the baseball lands, we are looking for the *horizontal* displacement. Remember, gravity will bring the ball back down, but will have *no* effect on the horizontal displacement. So we plug 2.8 into $x(t)$, giving

$$x(2.8) = 7.5(2.8) = 21.$$

   Thus, the baseball lands 21 m away from where you threw it.

3. To show that the baseball's trajectory is a parabola, we need to use the displacement equations,

$$y(t) = -4.9t^2 + 13t + 2, \quad x(t) = 7.5t,$$

and eliminate the variable t. This is not hard to do, since dividing the second equation by 7.5 gives $t = \frac{x}{7.5}$. Plugging back into the first equation:

$$\begin{aligned} y(t) &= -4.9t^2 + 13t + 2 \\ y &= -4.9\left(\frac{x}{7.5}\right)^2 + 13\left(\frac{x}{7.5}\right) + 2 \\ y &= -0.087x^2 + 1.73x + 2 \end{aligned}$$

Since the coefficient of x^2 is negative, this is the equation of a parabola which opens down.

Example 4

Suppose we want to create a circular fountain like in the video, where water shoots from spouts on the edge of the circle at ground level, and they all end up splashing in the center. We would like the spouts to shoot water at a 55° angle, and the diameter of the fountain is 50 m. (1) How fast does the water have to be shot out of the spouts for the waterspouts to converge in the center? (2) How high does the water go? Ignore any air resistance in this problem.

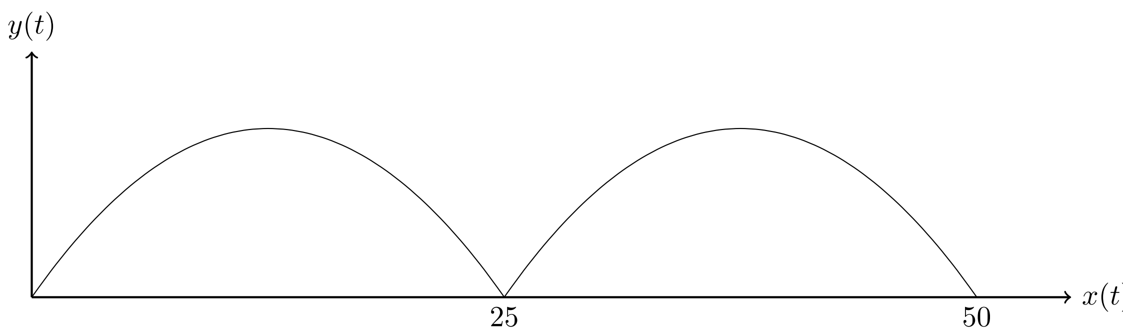


Figure 9.4: Planning converging fountains.

Let's rewrite the displacement equations for reference.

$$y(t) = -4.9t^2 + v_0 t + s_0, \quad x(t) = v_h t.$$

Like before, we need to find v_0, s_0, and v_h. Since the spouts are on the ground, we know that $s_0 = 0$.

Now let's draw what is called a **force diagram** in physics, like we did before. Here, we don't know the velocity, so we represent it by v. Reading off the right triangle, we have

$$\sin 55^\circ = \frac{v_0}{v}, \quad \cos 55^\circ = \frac{v_h}{v}.$$

Thus,

$$v_0 = v \sin 55^\circ \approx 0.82v, \quad v_h = v \cos 55^\circ \approx 0.57v.$$

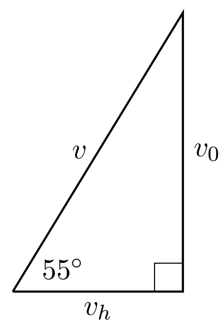


Figure 9.5: Decomposing a force into vertical and horizontal components.

Substituting back into the displacement equations (remember that $s_0 = 0$), we have

$$y(t) = -4.9t^2 + 0.82vt, \quad x(t) = 0.57vt.$$

Remember, v is a constant here.

Let's take stock of what we have. Here's the information we haven't used yet: the fact that the water has to travel 25 m in the horizontal direction to make it to the center, since the circle has a diameter of 50 m, and the fact that when the water *does* make it to the center, $y(t) = 0$.

First, we'll look at $y(t) = 0$. Note that because $s_0 = 0$, we won't need the quadratic formula.

$$\begin{aligned} y(t) &= 0 \\ -4.9t^2 + 0.82vt &= 0 \\ t(0.82v - 4.9t) &= 0 \\ t &= 0 \\ 0.82v - 4.9t &= 0 \\ 0.82v &= 4.9t \\ t &= \frac{0.82v}{4.9} \\ &\approx 0.17v \end{aligned}$$

Which value of t do we choose? The value $t = 0$ corresponds to the fact that the water shoots off from ground level, so we want $t = 0.17v$. This means once we figure out what v is, we can tell how long it takes the water to hit the center of the fountain.

So we use the last piece of information: the diameter of the fountain is 50 m. That means that at $t = 0.17v$ – when the water hits the center – it has traveled 25 m. We say this algebraically as

$$x(t) = x(0.17v) = 25.$$

Let's solve.

$$\begin{aligned} x(0.17v) &= 25 \\ (0.57v)(0.17v) &= 25 \\ 0.097v^2 &= 25 \\ v^2 &= \frac{25}{0.97} \\ v &= \sqrt{\frac{25}{0.97}} \\ &\approx 16.1 \end{aligned}$$

Therefore, we need the water to be shot out at 16.1 m/s (about 36 mph) so that it hits the center of the fountain exactly. Note that we choose the *positive* square root since the water is being shot *up* at an angle of 55°.

How high does the water go? Remember that when we know where the parabola opening down crosses the x-axis, the highest point occurs at the midpoint of those crossings. In other words, the water is at its highest point at a horizontal distance of $\frac{25}{2} = 12.5$ m.

But $y(t)$ is a function of t. So we *cannot* plug in 12.5 for t, since t is measured in *seconds,* not meters. So we have to go back to our equation for $x(t)$, solving $x(t) = 12.5$.

$$\begin{aligned} x(t) &= 0.57vt \\ &= (0.57)(16.1)t \\ &\approx 9.18t \\ 9.18t &= 12.5 \\ t &= \frac{12.5}{9.18} \\ &\approx 1.36 \end{aligned}$$

So the waterspouts reach their highest point at about 1.36 s. Plugging back into $y(t)$, we have

$$\begin{aligned} y(t) &= -4.9t^2 + 0.82vt \\ y(1.36) &= -4.9(1.36)^2 + 0.82(16.1)(1.36) \\ &\approx 8.9 \end{aligned}$$

So the highest the water goes is about 8.9 m, which is about 29 ft.

Homework

1. You are standing on the roof of a building which is 30 m tall. You accidentally drop your phone from the roof. How long will it take to hit the ground?

2. You are standing on the roof of a building which is 30 m tall. You throw a marble *up* from the roof at 12 m/s. (1) How long will it take to hit the ground? (2) At what velocity does it hit the ground?

3. Suppose you throw a baseball at an angle of $50°$ from the horizontal at a speed of 20 m/s. When the baseball leaves your hand, it is 2 m above the ground. (1) When will it hit the ground again? (2) How far away will the ball land? (3) By finding an equation in x and y, show that the trajectory the baseball takes is a parabola.

Solutions

1. We begin with the Displacement Equations, using $v_0 = 0$ and $s_0 = 30$:

$$s(t) = -4.9t^2 + 30.$$

When your phone hits the ground, $s(t) = 0$. Solving,

$$\begin{aligned} s(t) &= 0 \\ -4.9t^2 + 30 &= 0 \\ 4.9t^2 &= 30 \\ t^2 &\approx 6.12 \\ t &\approx 2.47 \end{aligned}$$

Thus, your phone hits the ground after about 2.47 s. Note we only considered the positive square root as we are looking for a time.

2. We begin with the Displacement Equations, using $v_0 = 12$ and $s_0 = 30$:

$$v(t) = -9.8t + 12, \quad s(t) = -4.9t^2 + 12t + 30.$$

(a) To find out how long the marble will take to hit the ground, we solve $s(t) = 0$ using the quadratic formula.

$$\begin{aligned} t &= \frac{-12 \pm \sqrt{12^2 - 4(-4.9)(30)}}{2(-4.9)} \\ &= \frac{-12 \pm \sqrt{144 + 588}}{-9.8} \\ &= \frac{-12 \pm 27.06}{-9.8} \\ \frac{-12 + 27.06}{-9.8} &\approx -1.5 \\ \frac{-12 - 27.06}{-9.8} &\approx 4.0 \end{aligned}$$

As t represents a time, we choose the positive value. Thus, the marble hits the ground after about 4.0 s.

(b) To find the velocity when the marble hits the ground, we substitute the value of t into the velocity equation, $v(t)$. Thus,

$$\begin{aligned} v(t) &= -9.8t + 12 \\ v(4.0) &= -9.8(4.0) + 12 \\ &\approx -27.2 \end{aligned}$$

So the marble hits the ground at about -27.2 m/s.

3. We know the formula for the vertical and horizontal displacement from our previous work:

$$y(t) = -4.9t^2 + v_0 t + s_0, \quad x(t) = v_h t.$$

We are given that $s_0 = 2$, but we will need to decompose the initial velocity vector in order to find v_0 and v_h.

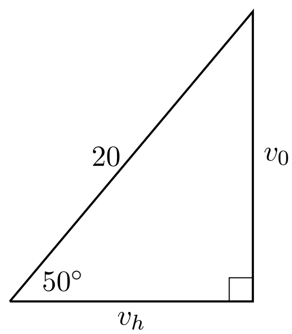


Figure 9.6: Decomposing a vector into vertical and horizontal components.

Reading off the triangle, we have

$$\sin 50° = \frac{v_0}{20}, \quad \cos 50° = \frac{v_h}{20}.$$

Then we get

$$v_0 = 20\sin 50° \approx 15.3, \quad v_h = 20\cos 50° \approx 12.9.$$

Thus, the vertical component of the velocity is about 15.3 m/s, and the horizontal component of the velocity is 12.9 m/s. So now we have equations for vertical and horizontal displacement:

$$y(t) = -4.9t^2 + 15.3t + 2, \quad x(t) = 12.9t.$$

(a) To see when the baseball will hit the ground, we solve $y(t) = 0$, since $y(t)$ is the vertical displacement:

$$-4.9t^2 + 15.3t + 2 = 0.$$

As before, we use the quadratic formula and choose the positive solution, which is $t \approx 3.2$ s.

(b) To see how far away the baseball lands, we are looking for the horizontal displacement. So we plug 3.2 into $x(t)$, giving

$$x(3.2) = 12.9(3.2) \approx 41.3.$$

Thus, the baseball lands 41.3 m away from where you threw it.

(c) To show that the baseball's trajectory is a parabola, we need to use the displacement equations,

$$y(t) = -4.9t^2 + 15.3t + 2, \quad x(t) = 12.9t,$$

and eliminate the variable t. Start by dividing the second equation by 12.9, giving $t = \frac{x}{12.9}$. Plugging back into the first equation:

$$\begin{aligned} y(t) &= -4.9t^2 + 15.3t + 2 \\ y &= -4.9\left(\frac{x}{12.9}\right)^2 + 15.3\left(\frac{x}{12.9}\right) + 2 \\ y &= -0.029x^2 + 1.19x + 2 \end{aligned}$$

Since the coefficient of x^2 is negative, this is the equation of a parabola which opens down.

9.2 Antiderivatives, II

We just learned how we can use the process of antidifferentiation to solve everyday problems in physics. So far, we've found the displacement from the velocity when velocity is a linear function. It's time to go a bit further.

First, some common notation. We said the most general antiderivative of -9.8 was $-9.8t + C$, where the variable was t (for time), and C was whatever constant we chose. We write this as

$$\int -9.8\, dt = -9.8t + C.$$

The "$\int$" is the notation for taking an antiderivative, and the "dt" means the variable is t. If we were working with the variable were x, we would write

$$\int -9.8\, dx = -9.8x + C.$$

And instead of always saying "the most general antiderivative of $f(x)$," we just write

$$\int f(x)\, dx.$$

Learning to think backwards about differentiation does take a lot of practice.

Examples

We'll work out several short examples. It might be a good idea to have page 240 handy.

1. Find $\displaystyle\int x^3\, dx$.

   We know that when using the Power Rule to differentiate, we *decrease* the exponent by 1. So to antidifferentiate, we need to *increase* the exponent by one. But we don't just get x^4, because $\dfrac{d}{dx}x^4 = 4x^3$ – we have an extra factor of 4. So we compensate by dividing by 4, giving
   $$\int x^3\, dx = \frac{1}{4}x^4 + C.$$
   We can check by differentiating:
   $$\frac{d}{dx}\left(\frac{1}{4}x^4 + C\right) = \frac{1}{4}(4x^3) + 0 = x^3.$$

2. Find $\displaystyle\int x^n\, dx$.

   We take the same approach as in the previous problem. Increasing the exponent by 1 gives x^{n+1}, but
   $$\frac{d}{dx}x^{n+1} = (n+1)x^n.$$

So we get an extra factor of $n+1$, which we compensate for by dividing. Thus,

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C.$$

Again, we can check by differentiating:

$$\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + C\right) = \frac{1}{n+1}(n+1)x^n + 0 = x^n.$$

This rule is so important, we box it.

Inverse Power Rule

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1.$$

It is important to note why we must have $n \neq -1$. This is because we would get $\frac{1}{0}x^0$, which is undefined. But when $n = -1$, we have

$$\begin{aligned}\int x^{-1}\,dx &= \int \frac{1}{x}\,dx \\ &= \ln x + C,\end{aligned}$$

since we know that $\frac{d}{dx}\ln x = \frac{1}{x}$.

3. Find $\int (x^4 - 2x^3 + 5)\,dx$.

   We apply the Inverse Power Rule.

$$\begin{aligned}\int (x^4 - 2x^3 + 5)\,dx &= \frac{1}{5}x^5 - 2\left(\frac{1}{4}x^4\right) + 5x + C \\ &= \frac{1}{5}x^5 - \frac{1}{2}x^4 + 5x + C.\end{aligned}$$

4. Find $\int \sqrt{x}\,dx$.

   As we did with derivatives, we rewrite as a power and then use the Inverse Power Rule.

$$\begin{aligned}\int \sqrt{x}\,dx &= \int x^{1/2}\,dx \\ &= \frac{1}{3/2}x^{3/2} + C \\ &= \frac{2}{3}x^{3/2} + C.\end{aligned}$$

5. Find $\int \frac{1}{x^6}\,dx$.

   Again, we must rewrite, as we needed to do with derivatives.

$$\begin{aligned}\int \frac{1}{x^6}\,dx &= \int x^{-6}\,dx \\ &= \frac{1}{-5}x^{-5} + C \\ &= -\frac{1}{5x^5} + C.\end{aligned}$$

   Be careful when *adding* 1 to negative exponents.

6. Find $\int \cos(x)\,dx$.

   Since $\frac{d}{dx}\sin(x) = \cos(x)$, then

$$\int \cos(x)\,dx = \sin(x) + C.$$

7. Find $\int \sin(x)\,dx$.

   The answer isn't $\cos(x) + C$, since $\frac{d}{dx}\cos(x) = -\sin(x)$. We have to compensate by putting in a negative sign.

$$\int \sin(x)\,dx = -\cos(x) + C.$$

8. Find $\int \frac{3}{x^2+1}\,dx$.

   We recognize $\frac{1}{x^2+1}$ as the derivative of $\arctan(x)$, so that

$$\int \frac{3}{x^2+1}\,dx = 3\arctan(x) + C.$$

9. Find $\int \frac{2}{\sqrt{1-x^2}}\,dx$.

   We recognize the derivative of $\arcsin(x)$ here.

$$\int \frac{2}{\sqrt{1-x^2}}\,dx = 2\arcsin(x) + C.$$

Initial Value Problems

We've already seen initial value problems, as when working with physics. In one example, we threw a marble down at 10 m/s. We used the fact that $a(t) = -9.8$, worked backwards (that is, took the antiderivative), and used the 10 to find $C = v_0$. Now the derivative of velocity is acceleration, and so $v'(t) = a(t)$. Let's redo this problem using our new notation.

We can rewrite this problem as follows.

Solve the initial value problem $v'(t) = -9.8,\ v(0) = 10.$

Here is the solution using new notation. There are no new steps or concepts involved here, just a different way of stating the problem (as an initial value problem) and writing the solution (using antiderivative notation).

$$\begin{aligned} v(t) &= \int v'(t)\, dt \\ &= \int -9.8\, dt \\ &= -9.8t + C. \\ v(0) &= -9.8(0) + C \\ &= 10. \\ C &= 10. \\ v(t) &= -9.8t + 10. \end{aligned}$$

Essentially, an initial value problem presents you with a derivative, but also some value of the function you're looking for. This additional information will allow you to find the $+C$.

Now let's look at some examples.

10. Solve the initial value problem $f'(x) = x^2 + 2x + 1$, $f(3) = 15$. We first take an antiderivative.

$$\begin{aligned} f(x) &= \int (x^2 + 2x + 1)\, dx \\ &= \frac{1}{3}x^3 + 2\left(\frac{1}{2}x^2\right) + x + C \\ &= \frac{1}{3}x^3 + x^2 + x + C. \end{aligned}$$

Then

$$\begin{aligned} f(3) &= \frac{1}{3} \cdot 3^3 + 3^2 + 3 + C \\ &= 9 + 9 + 3 + C \\ &= 21 + C = 15, \end{aligned}$$

Thus, $C = -6$, and so $f(x) = \frac{1}{3}x^3 + x^2 + x - 6$.

11. Solve the initial value problem $f'(x) = 3^x$, $f(0) = 1$.

    Here, we need to remember that $\frac{d}{dx}3^x = 3^x \ln 3$, so to get a derivative of just 3^x, we need to divide by $\ln 3$.

    $$f(x) = \int 3^x \, dx = \frac{3^x}{\ln 3} + C.$$

    Let's confirm that dividing by $\ln 3$ was the right move.

    $$\begin{aligned} \frac{d}{dx}\left(\frac{3^x}{\ln 3} + C\right) &= \frac{1}{\ln 3}(3^x \ln 3) + 0 \\ &= 3^x. \end{aligned}$$

    Now we use the given fact $f(0) = 1$ to find C.

    $$\begin{aligned} f(0) &= 1 \\ \frac{3^0}{\ln 3} + C &= 1 \\ \frac{1}{\ln 3} + C &= 1 \\ C &= 1 - \frac{1}{\ln 3} \end{aligned}$$

    Thus, $f(x) = \frac{3^x}{\ln 3} + 1 - \frac{1}{\ln 3}$.

12. Solve the initial value problem $f'(x) = \sec^2(x) + \sin(x)$, $f(\pi/4) = 1$.

    We need to remember that $\frac{d}{dx}\tan(x) = \sec^2(x)$. Then

    $$\int (\sec^2(x) + \sin(x)) \, dx = \tan(x) - \cos(x) + C.$$

    Therefore,

    $$\begin{aligned} f(\pi/4) &= 1 \\ \tan(\pi/4) - \cos(\pi/4) + C &= 1 \\ 1 - \frac{1}{\sqrt{2}} + C &= 1 \\ C &= \frac{1}{\sqrt{2}} \end{aligned}$$

    So we get $f(x) = \tan(x) - \cos(x) + \frac{1}{\sqrt{2}}$.

As we saw with projectile motion – a common example in physics – we knew the acceleration, and used antidifferentiation to find the displacement. Let's return to a previous example, where we throw down a marble off a 20 meter high roof at 10 m/s. Restated as an initial value problem, we have

$$\text{Solve the initial value problem } s''(t) = -9.8,\ s'(0) = -10,\ s(0) = 20.$$

Since $s(t)$ is the displacement, $s''(t)$ is the acceleration, which is constant. Since the velocity is $s'(t)$, the statement $s'(0) = -10$ means that we are throwing the marble down at 10 m/s. And $s(0) = 20$ means we are throwing it from a height of 20 m. So the entire problem is restated using $s(t)$ *only.* This is the way such problems are usually stated in physics.

13. Solve the initial value problem $s''(t) = t^4 - t^2$, $s'(0) = 5$, $s(0) = 10$.

    Since we are given a *second* derivative, we have to antidifferentiate twice – first to find $s'(t)$, and then to find $s(t)$.

    $$s'(t) = \int (t^4 - t^2)\,dt = \frac{1}{5}t^5 - \frac{1}{3}t^3 + C.$$

    We use the information $s'(0) = 5$ to find C.

    $$5 = s'(0) = \frac{1}{5} \cdot 0^5 - \frac{1}{3} \cdot 0^3 + C,$$

    so that $C = 5$ and $s'(t) = \frac{1}{5}t^5 - \frac{1}{3}t^3 + 5$. We now antidifferentiate again to find $s(t)$.

    $$\begin{aligned} s(t) &= \int \left(\frac{1}{5}t^5 - \frac{1}{3}t^3 + 5\right) dt \\ &= \frac{1}{5}\left(\frac{1}{6}t^6\right) - \frac{1}{3}\left(\frac{1}{4}t^4\right) + 5t + C \\ &= \frac{1}{30}t^6 - \frac{1}{12}t^4 + 5t + C. \end{aligned}$$

    It is clear that plugging 0 into $s(t)$ just gives back C, so $C = 10$ from the given information $s(0) = 10$. Thus, $s(t) = \frac{1}{30}t^6 - \frac{1}{12}t^4 + 5t + 10$.

No single step in any of these problems was especially tricky. What makes this section challenging is that you have to remember *all* of your derivative formulas. And because there are many small steps, you have to really pay attention to the algebra. There are a few mistakes commonly made, so I'll make a short list here.

1. Incorrectly rewriting as powers of x, as in $\sqrt{x} = x^{1/2}$ and $\frac{1}{x^3} = x^{-3}$,
2. Using the Inverse Power Rule with $\frac{1}{x} = x^{-1}$, since $n = -1$ is not allowed. Instead, notice that $\frac{1}{x}$ is the derivative of $\ln x$,
3. Using the wrong $+$ or $-$ when taking the antiderivatives of $\sin(x)$ and $\cos(x)$,
4. Making a calculation error when using initial values to find C.

Finding Antiderivatives

Here is a summary of all the antiderivatives we know (that is, you can just use them at any time without justification), and the basic rules of antidifferentiation.

1. $\displaystyle\int \cos(x)\,dx = \sin(x).$
2. $\displaystyle\int \sin(x)\,dx = -\cos(x) + C.$
3. $\displaystyle\int \sec^2(x)\,dx = \tan(x) + C.$
4. $\displaystyle\int e^x\,dx = e^x + C.$
5. $\displaystyle\int \frac{1}{x}\,dx = \ln x + C.$
6. $\displaystyle\int b^x\,dx = \frac{b^x}{\ln b} + C.$
7. $\displaystyle\int \frac{1}{\sqrt{1-x^2}} = \arcsin(x) + C.$
8. $\displaystyle\int \frac{1}{x^2+1} = \arctan(x) + C.$
9. The Inverse Power Rule:
   When $n \neq -1$,
   $$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C.$$
10. The Sum Rule:
    $$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx + C.$$
11. The Difference Rule:
    $$\int (f(x)-g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx + C.$$
12. The Constant Multiple Rule:
    $$\int cf(x)\,dx = c\int f(x)\,dx + C.$$

The Inverse Chain Rule: To integrate $\displaystyle\int f'(g(x))g'(x)\,dx$:

1. Look for a $g(x)$ and $g'(x)$ pair in the integrand – $g'(x)$ can be off by a constant multiple;
2. If $g'(x)$ is off by a constant multiple, multiply and divide by this constant and factor out;
3. Substitute $u = g(x)$, and solve for $du = g'(x)\,dx$;
4. Rewrite the integral in terms of u – all x's should disappear;
5. Find the antiderivative with respect to u;
6. Substitute back to rewrite in terms of x only.

Homework

1. Find $\int (x^5 - 6x^3 + x^2 - 4)\,dx$.
2. Find $\int \frac{1}{\sqrt{x}}\,dx$.
3. Find $\int \left(\frac{1}{x} + \frac{1}{x^2}\right)\,dx$.
4. Find $\int (3\sin(x) - 5\cos(x))\,dx$.
5. Find $\int \frac{6}{\sqrt{1-x^2}}\,dx$.
6. Solve the initial value problem $f'(x) = x^3 - 3x^2 - 1,\ f(2) = 10$.
7. Solve the initial value problem $f'(x) = \cos(x) - \sec^2(x),\ f(\pi/3) = -\sqrt{3}$.
8. Solve the initial value problem $s''(t) = t^3 + t,\ s'(0) = 3,\ s(0) = 8$.

Solutions

1.

$$\begin{aligned}\int (x^5 - 6x^3 + x^2 - 4)\,dx &= \frac{1}{6}x^6 - 6\left(\frac{1}{4}x^4\right) + \frac{1}{3}x^3 - 4x + C \\ &= \frac{1}{6}x^6 - \frac{3}{2}x^4 + \frac{1}{3}x^3 - 4x + C\end{aligned}$$

2.

$$\begin{aligned}\int \frac{1}{\sqrt{x}}\,dx &= \int x^{-1/2}\,dx \\ &= \frac{1}{1/2}x^{1/2} + C \\ &= 2\sqrt{x} + C.\end{aligned}$$

3.

$$\begin{aligned}\int \left(\frac{1}{x} + \frac{1}{x^2}\right)\,dx &= \int (x^{-1} + x^{-2})\,dx \\ &= \ln x + \frac{1}{-1}x^{-1} + C \\ &= \ln x - \frac{1}{x} + C.\end{aligned}$$

4.

$$\begin{aligned}\int (3\sin(x) - 5\cos(x))\,dx &= 3(-\cos(x)) - 5\sin(x) + C \\ &= -3\cos(x) - 5\sin(x) + C.\end{aligned}$$

5.

$$\int \frac{6}{\sqrt{1-x^2}}\,dx = 6\arcsin(x) + C.$$

6. First, find a general antiderivative for $f'(x)$.

$$\begin{aligned}f(x) &= \int f'(x)\,dx \\ &= \int (x^3 - 3x^2 - 1)\,dx \\ &= \frac{1}{4}x^4 - 3\left(\frac{1}{3}x^3\right) - x + C \\ &= \frac{1}{4}x^4 - x^3 - x + C\end{aligned}$$

Then find C.

$$\begin{aligned} 10 &= f(2) \\ &= \frac{1}{4} \cdot 2^4 - 2^3 - 2 + C \\ &= 4 - 8 - 2 + C \\ &= -6 + C \\ C &= 6. \end{aligned}$$

Therefore,

$$f(x) = \frac{1}{4}x^4 - x^3 - x + 6.$$

7. First, find a general antiderivative for $f'(x)$.

$$\begin{aligned} f(x) &= \int f'(x)\,dx \\ &= \int (\cos(x) - \sec^2(x))\,dx \\ &= \sin(x) - \tan(x) + C. \end{aligned}$$

Next, find C.

$$\begin{aligned} -\sqrt{3} &= f(\pi/3) \\ &= \sin(\pi/3) - \tan(\pi/3) + C \\ &= \frac{\sqrt{3}}{2} - \sqrt{3} + C \\ C &= -\frac{\sqrt{3}}{2}. \end{aligned}$$

Thus,

$$f(x) = \sin(x) - \tan(x) - \frac{\sqrt{3}}{2}.$$

8. First, find $s'(t)$.

$$\begin{aligned} s'(t) &= \int s''(t)\,dt \\ &= \int (t^3 + t)\,dt \\ &= \frac{1}{4}t^4 + \frac{1}{2}t^2 + C. \end{aligned}$$

Now use $s'(0) = 3$ to find C.

$$\begin{aligned} 3 &= \frac{1}{4} \cdot 0^4 + \frac{1}{2} \cdot 0^2 + C \\ C &= 3. \end{aligned}$$

Thus,

$$s'(t) = \frac{1}{4}t^4 + \frac{1}{2}t^2 + 3.$$

Now that we have $s'(t)$, we can find $s(t)$.

$$\begin{aligned} s(t) &= \int s'(t)\,dt \\ &= \int \left(\frac{1}{4}t^4 + \frac{1}{2}t^2 + 3\right) dt \\ &= \frac{1}{4} \cdot \frac{1}{5}t^5 + \frac{1}{2} \cdot \frac{1}{3}t^3 + 3t + C \\ &= \frac{1}{20}t^5 + \frac{1}{6}t^3 + 3t + C. \end{aligned}$$

Finally, use $s(0) = 8$ to find C.

$$\begin{aligned} 8 &= s(0) \\ &= \frac{1}{20} \cdot 0^5 + \frac{1}{6} \cdot 0^3 + 3 \cdot 0 + C \\ C &= 8. \end{aligned}$$

So our final answer is

$$s(t) = \frac{1}{20}t^5 + \frac{1}{6}t^3 + 3t + 8.$$

9.3 Areas

We've come around full circle at this point. Let's review an earlier example, illustrated in Figure 9.7.

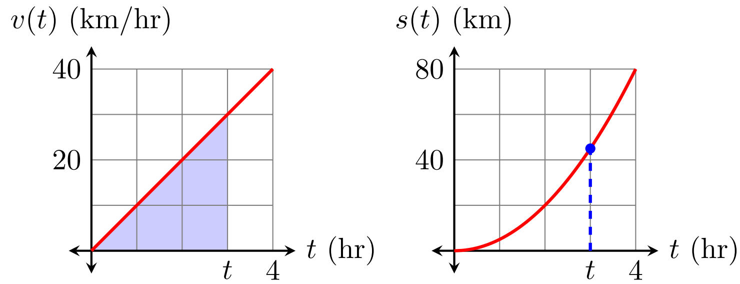
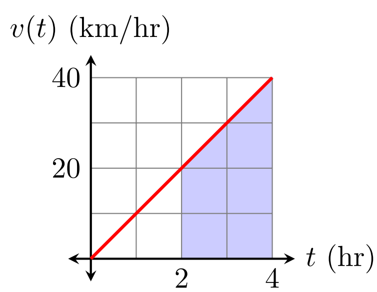


Figure 9.7: Velocity graph (left), and displacement graph (right).

Here, the velocity is given by $v(t) = 10t$ since it is linear, and at $t = 4$, we are at 40 km/hr. The displacement $s(t)$ up to time t is the area under the velocity curve up to time t. By looking at the areas of triangles, we found out that $s(t) = 5t^2$. If you look closely, you'll observe that $10t$ is the derivative of $5t^2$, and $5t^2$ is an antiderivative of $10t$, since

$$\begin{aligned}\int 10t\,dt &= 10\left(\frac{1}{2}t^2\right) + C\\ &= 5t^2 + C.\end{aligned}$$

Now let's ask the following question: How far have we driven between time $t = 2$ and $t = 4$? We could shade in the following area.



We could calculate this using the formula for the area of a trapezoid. But it turns out that there is a simpler way which is very important in calculus. We know that $s(t)$ measures the area under the velocity curve, but *starting at* $t = 0$. What do we do when we start at $t = 2$?

The key is to look at the blue trapezoid as the *difference* of two areas, as shown in Figure 9.8. Most labels have been removed to make the geometry easier to see.

As you can see, if you take the area of the large triangle, A_2, and subtract off the area of the small triangle, A_3, you get the area of the trapezoid, which is the distance traveled between time $t = 2$ and $t = 4$. Take a moment to really understand this from the figures.

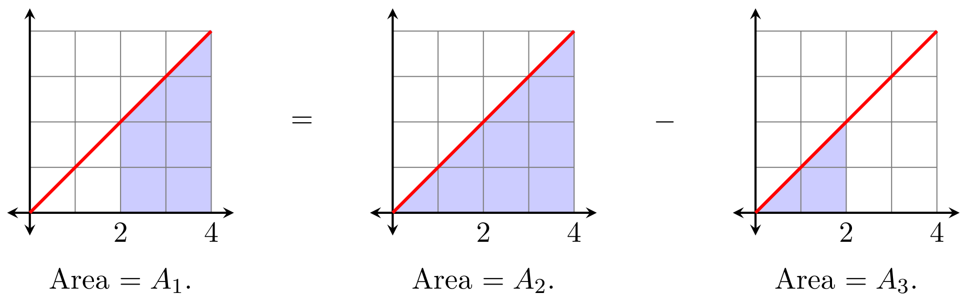


Figure 9.8: One area as the difference of two others.

Why is this significant? Because the areas A_2 and A_3 have their left endpoints at $t = 0$. This means that $A_2 = s(4)$, the displacement from time $t = 0$ to time $t = 4$, and $A_3 = s(2)$. But we know what $s(t)$ is, so we can compute the area A_1.

$$
\begin{aligned}
s(t) &= 5t^2 \\
A_1 &= A_2 - A_3 \\
&= s(4) - s(2) \\
&= 5(4^2) - 5(2^2) \\
&= 80 - 20 \\
&= 60.
\end{aligned}
$$

Of course there is nothing special about $t = 2$ and $t = 4$, so we could say that the area under the velocity curve from $t = a$ to $t = b$ is just $s(b) - s(a)$. The same logic applies.

Now let's write this using antiderivative notation. We know that the displacement is the area under the velocity curve, but it is *also* an antiderivative of the velocity. So we write

$$\int_a^b v(t)\,dt = s(b) - s(a),$$

which we read as "the area under the velocity curve from time $t = a$ to $t = b$ is equal to $s(b) - s(a)$." This is what we just observed, but we are using our new notation to describe it.

Thus we have the following geometric interpretation of derivatives and antiderivatives.

The...	is used for...
derivative	finding slopes of tangent lines.
antiderivative	finding areas under curves.

We observed this at the very beginning, but now we have developed calculus tools to find slopes and areas. It was easy to do this using formulas from geometry when we only considered constant or linear velocities, but now we can find slopes and areas for a very large group of functions – functions where there are no simple geometrical formulas to aid us.

By considering velocity and displacement, we were able to write

$$\int_a^b v(t)\,dt = s(b) - s(a).$$

But a similar statement can be made for other functions. The important point is that $s(t)$ is an antiderivative of $v(t)$. That's all we needed to make this work. Let's restate this in terms of x, since that's how it's usually stated.

Fundamental Theorem of Calculus, Part I

Let $f(x)$ be given, and suppose that $F(x)$ is an antiderivative of $f(x)$. Then for a and b in the domain of $f(x)$,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

A word on notation. An antiderivative written in the form

$$\int f(x)\,dx$$

is called an **indefinite integral,** and an antiderivative written in the form

$$\int_a^b f(x)\,dx$$

is called a **definite integral.** It is "definite" since you are specifying the interval $[a, b]$, so when no interval is specified, it is "indefinite."

Very often, the word "integrate" is used to mean the same thing as "antidiffereniate," and "integral" is used for "antiderivative."

Example 1

Find the area underneath $f(x) = \sin(x)$ and above the x-axis on the interval $[0, \pi]$, as shown in Figure 9.17. A good first step is to make a reasonable guess at the area. Looking at the right in

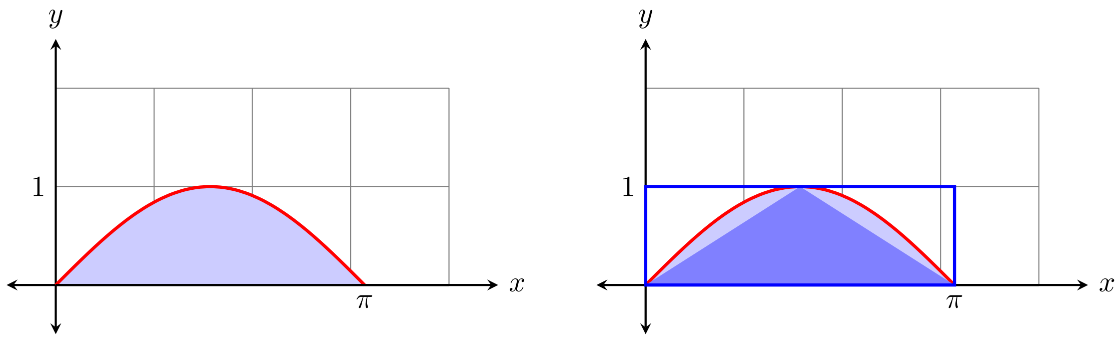


Figure 9.9: Area under $f(x) = \sin(x)$.

Figure 9.17, you can see a dark blue triangle which fits inside the area, which has area

$$\frac{1}{2}(\pi)(1) \approx 1.57.$$

Further, the area sits inside the blue rectangle, which has area $\pi \approx 3.14$. So the total area under $\sin(x)$ is between 1.57 and 3.14, but closer to 1.57.

Now let's use calculus. We use The Fundamental Theorem of Calculus with $f(x) = \sin(x)$, $a = 0$, and $b = \pi$. Then

$$\int_0^\pi \sin(x)\,dx = F(\pi) - F(0),$$

where $F(x)$ is any antiderivative of $\sin(x)$. But we know that $-\cos(x)$ is an antiderivative of $\sin(x)$, and so

$$\begin{aligned}\int_0^\pi \sin(x)\,dx &= -\cos(\pi) - (-\cos(0)) \\ &= -(-1) - (-1) \\ &= 2.\end{aligned}$$

This fits well with our guesstimate, so we can be confident in our answer. The answer turned out to be very simple, but it is important to note that there is no simple geometrical formula we could have used to get 2. Calculus is really needed here, as it is for calculating most areas.

Example 2

Find the area of the shaded regions bounded by $f(x) = \sin(x)$ on the interval $[0, 2\pi]$, as shown in Figure 9.10.

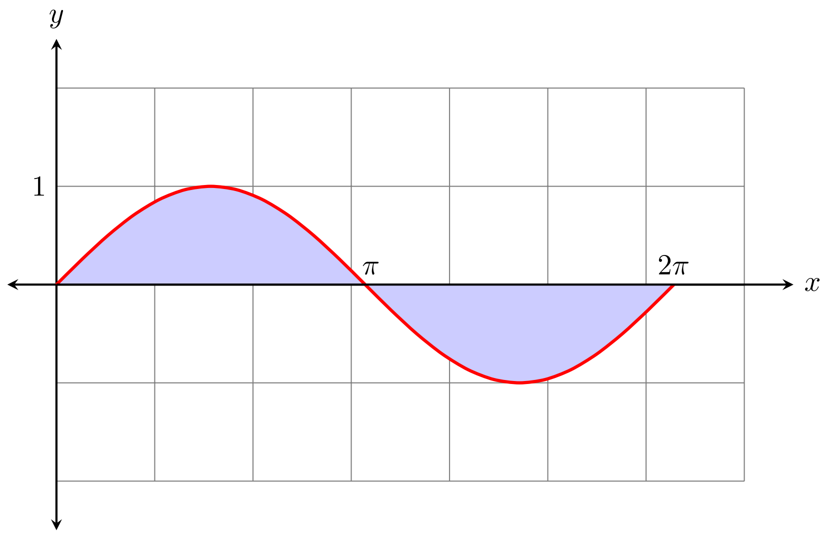


Figure 9.10: Area bounded by $f(x) = \sin(x)$.

We know from the previous problem that the answer should be 4, since there are two regions of area 2. But remember that regions *below* the x-axis make negative contributions to the area. So if we use the Fundamental Theorem of Calculus with $a = 0$ and $b = 2\pi$, we get

$$\begin{aligned}\int_0^{2\pi} \sin(x)\, dx &= -\cos(2\pi) - (-\cos(0)) \\ &= -1 - (-1) \\ &= 0.\end{aligned}$$

In other words, the areas cancel each other out, since there are two identically shaped regions, but one lies above the x-axis, and the other lies below the x-axis.

Example 3

Find the area below the curve $f(x) = 4 - x^2$ and above the x-axis, as shown in Figure 9.11.

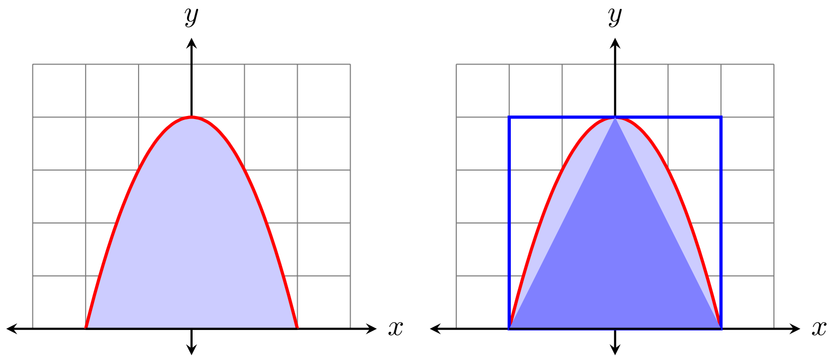


Figure 9.11: Area bounded by $f(x) = 4 - x^2$.

We can use the Fundamental Theorem of Calclulus here with $f(x) = 4 - x^2$. From the figure, it looks like we can use $a = -2$ and $b = 2$. To confirm this, we need to see where $f(x)$ crosses the x-axis. Solving $f(x) = 4 - x^2 = 0$ gives us $x = \pm 2$.

We can make a few guesstimates by looking at the right of Figure 9.11. The area is larger than the area of the blue triangle, which is $(4 \cdot 4)/2 = 8$, but smaller than the area of the blue rectangle, which is 16. We can visually see that it should be closer to 8 than to 16. To use the Fundamental Theorem of Calculus, we need an antiderivative of $f(x) = 4 - x^2$. Using the Inverse Power Rule, we can use $F(x) = 4x - \frac{1}{3}x^3$. Then

$$\begin{aligned}
\int_{-2}^{2} (4 - x^2)\,dx &= F(2) - F(-2) \\
&= 4 \cdot 2 - \frac{1}{3} \cdot 2^3 - \left(4(-2) - \frac{1}{3}(-2)^3\right) \\
&= 8 - \frac{8}{3} - \left(-8 - \left(\frac{-8}{3}\right)\right) \\
&= 8 - \frac{8}{3} + 8 - \frac{8}{3} \\
&= 16 - \frac{16}{3} \\
&= \frac{48}{3} \\
&= \frac{32}{3} \\
&\approx 10.7
\end{aligned}$$

This is consistent with our guesstimates. Here, finding an antiderivative is the easy part. It takes several steps to evaluate $F(2) - F(-2)$, so you need to be very careful.

There is an alternative way to approach this problem which cuts down on the algebra. As you can see on the right of Figure 9.12, the region we're looking at is symmetrical, so we can find the area of *half* the region and then multiply by 2.

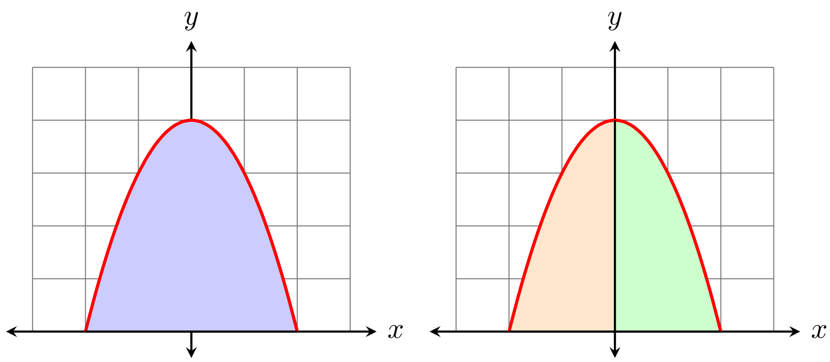


Figure 9.12: Using symmetry.

How will this help? The calculation is identical to the one we just made, except we use $a = 0$.

$$\begin{aligned}
\int_0^2 (4 - x^2)\,dx &= F(2) - F(0) \\
&= 4 \cdot 2 - \frac{1}{3} \cdot 2^3 - \left(4 \cdot 0 - \frac{1}{3} \cdot 0^3\right) \\
&= 8 - \frac{8}{3} - 0 \\
&= \frac{24}{3} - \frac{8}{3} \\
&= \frac{16}{3}
\end{aligned}$$

You can see how using $a = 0$ makes the calculations a lot simpler. Now we just need to double the area of one half of the region, and so the total area is $2 \cdot \frac{16}{3} = \frac{32}{3}$. As you can see, it is a good idea to take advantage of a symmetrical region so the calculations become easier.

Example 4

Find the area above the x-axis and below the curve $y = |x|$ on the interval $[-2, 3]$. See Figure 9.13.

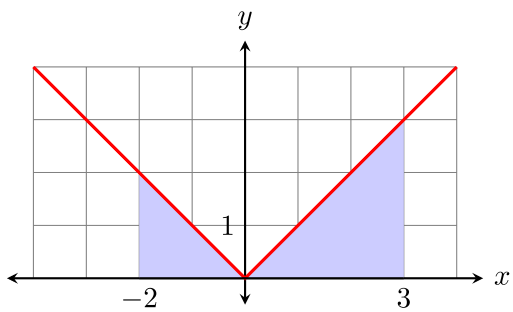


Figure 9.13: Area under $y = |x|$.

While we can just find the area of the triangles, let's see how using calculus would work. Thus, we want to find $\int_{-2}^{3} |x|\, dx$. The difficulty lies in finding an antiderivative of $|x|$. Since an antiderivative of x is $\frac{1}{2}x^2$, you might be tempted to choose $\left|\frac{1}{2}x^2\right|$. But since x^2 is always positive, then

$$\left|\frac{1}{2}x^2\right| = \frac{1}{2}x^2,$$

and so the derivative of $\left|\frac{1}{2}x^2\right|$ is x, *not* $|x|$.

So here, we have to go back to the piecewise definition of $y = |x|$:

$$y = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

This means we can break apart the integral into two separate parts, using the piecewise definition. Because we *do* know how to take antiderivatives of x and $-x$. And so

$$\int_{-2}^{3} |x|\, dx = \int_{-2}^{0} (-x)\, dx + \int_{0}^{3} x\, dx.$$

For the first integral, we use $F(x) = -\frac{1}{2}x^2$, so

$$\begin{aligned} \int_{-2}^{0} (-x)\, dx &= F(0) - F(-2) \\ &= -\frac{1}{2} \cdot 0^2 - \left(-\frac{1}{2}(-2)^2\right) \\ &= 0 - (-2) \\ &= 2. \end{aligned}$$

For the second integral, we use $F(x) = \frac{1}{2}x^2$, so

$$\begin{aligned}\int_0^3 x\,dx &= F(3) - F(0)\\ &= \frac{1}{2}\cdot 3^2 - \left(-\frac{1}{2}\cdot 0^2\right)\\ &= \frac{9}{2}.\end{aligned}$$

Putting it all together, we get

$$\begin{aligned}\int_{-2}^3 |x|\,dx &= \int_{-2}^0 (-x)\,dx + \int_0^3 x\,dx\\ &= 2 + \frac{9}{2}\\ &= \frac{4}{2} + \frac{9}{2}\\ &= \frac{13}{2}.\end{aligned}$$

What this example illustrates is that you can break up integrals if you have to. In other words, you can always use an intermediate point. Using integral notation, when $a \le b \le c$, then it is always the case that

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Example 5

Let's suppose we want to find the area *between* two curves, as shown in Figure 9.14. How would we do this?

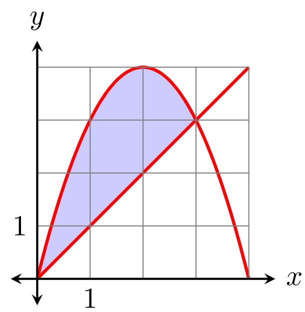


Figure 9.14: Area between the curves $f(x) = 4x - x^2$ and $f(x) = x$.

First, we need to find out where the curves intersect. In other words, we need to solve:

$$\begin{aligned} 4x - x^2 &= x \\ 3x - x^2 &= 0 \\ x(3 - x) &= 0 \\ x &= 0, 3 \end{aligned}$$

Now we use a technique similar to that used in Figure 9.8, and consider the area as a difference of two areas.

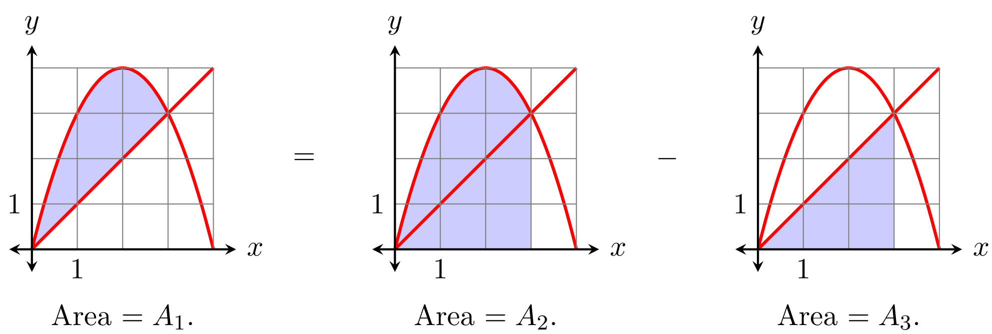


Figure 9.15: One area as the difference of two others.

Given the previous examples, and since we know that the curves intersect when $x = 0$ and $x = 3$, we know that $A_2 = \int_0^3 (4x - x^2)\, dx$ and $A_3 = \int_0^3 x\, dx$.

Therefore,

$$\begin{aligned}
A_1 &= A_2 - A_3 \\
&= \int_0^3 (4x - x^2)\,dx - \int_0^3 x\,dx \\
&= \int_0^3 (4x - x^2 - x)\,dx \\
&= \int_0^3 (3x - x^2)\,dx
\end{aligned}$$

First, we find an antiderivative for $3x - x^2$ using the Inverse Power Rule, which is

$$\begin{aligned}
F(x) &= 3\left(\frac{1}{2}x^2\right) - \frac{1}{3}x^3 \\
&= \frac{3}{2}x^2 - \frac{1}{3}x^3.
\end{aligned}$$

Then

$$\begin{aligned}
\int_0^3 (3x - x^2)\,dx &= F(3) - F(0) \\
&= \frac{3}{2}\cdot 3^2 - \frac{1}{3}\cdot 3^3 - \left(\frac{3}{2}\cdot 0^2 - \frac{1}{3}\cdot 0^3\right) \\
&= \frac{27}{2} - 9 \\
&= \frac{27}{2} - \frac{18}{2} \\
&= \frac{9}{2} \\
&= 4.5.
\end{aligned}$$

Counting square units of the area from the graph, this seems reasonable. Remember, you will always get the graph, so use it to check that your answer makes sense.

Homework

1. Consider the region below the curve $f(x) = \sin(x) + \cos(x) + 2$ and above the x-axis on the interval $[0, 2\pi]$.

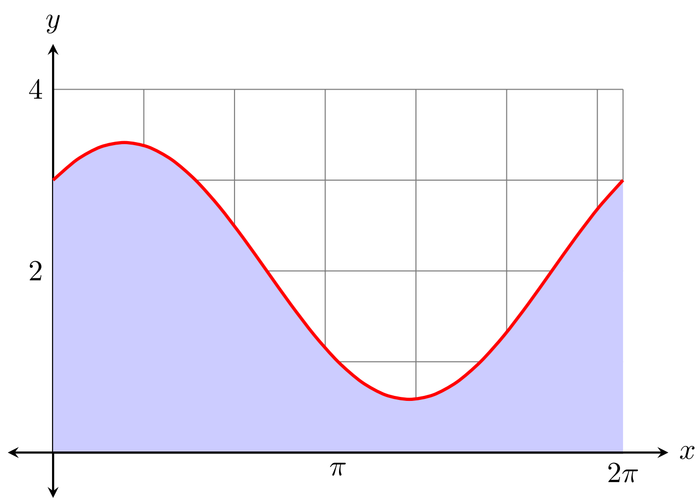


Figure 9.16: Area under $f(x) = \sin(x) + \cos(x) + 2$.

   (a) Visually, it looks like the area of this region is about half the area of the rectangular grid. Calculate this guesstimate for the area.

   (b) Using calculus, find the exact area. Is it close to your guesstimate?

2. Consider the function $f(x) = 16 - x^4$. Graph this on desmos. We will be looking at the area above the x-axis.

   (a) By drawing a triangle inside and a rectangle outside this region, find lower and upper guesstimates for the area. See Example 3.

   (b) By using symmetry appropriately, calculate the area of this region.

   (c) Verify that the area lies between your two guesstimates.

3. Consider the function defined below.

$$f(x) = \begin{cases} x+2, & x<1, \\ -x+4, & x \geq 1 \end{cases}$$

   (a) Using basic geometry formulas, find the area bounded by $f(x)$ and the x-axis on the interval $[-4, 3]$. Be careful about negative areas.

   (b) By writing the area as two separate integrals, compute the area using calculus.

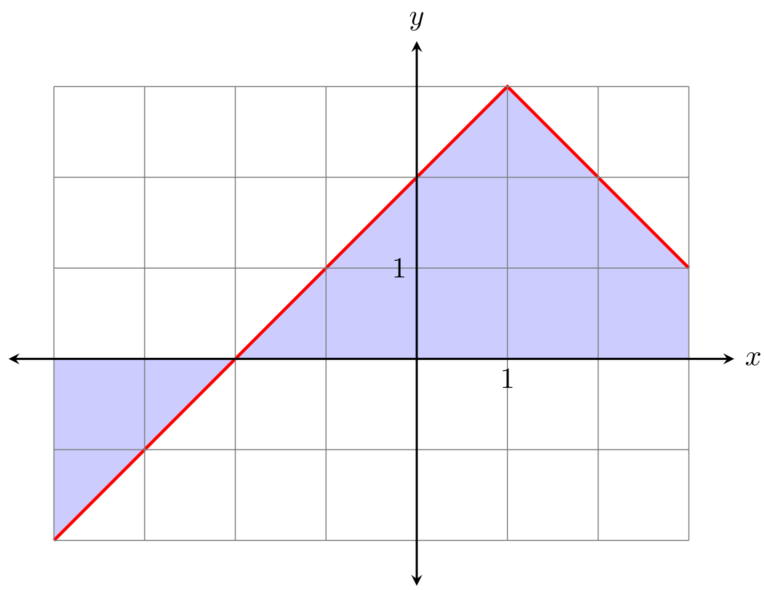


Figure 9.17: Area bounded by $f(x)$ and the x-axis.

4. Find the area between the curves $f(x) = x^2 - 4$ and $f(x) = 2 - x$. Graph using desmos to check your work.

5. Find the area between the curves $f(x) = 4x$ and $f(x) = x^3$ which lies in Quadrant I (that is, where x- and y-values are positive).

Solutions

1. (a) The rectangle has dimensions 4 by 2π, and so half of this is

$$\frac{1}{2} \cdot 4 \cdot 2\pi = 4\pi.$$

(b) An antiderivative for $f(x)$ is $-\cos(x) + \sin(x) + 2x$, so the area is

$$\begin{aligned}\int_0^{2\pi} (\sin(x) + \cos(x) + 2)\, dx &= (-\cos(2\pi) + \sin(2\pi) + 2(2\pi)) - (-\cos(0) + \sin(0) + 2(0)) \\ &= (-1 + 0 + 4\pi) - (-1 + 0 + 0) \\ &= 4\pi.\end{aligned}$$

2. (a) The graph crosses the x-axis at -2 and 2, which you get by solving $16 - x^4 = 0$. So the triangle has a base of 4 and a height of 16, so its area is

$$\frac{1}{2} \cdot 4 \cdot 16 = 32.$$

The rectangle has a base of 4 and a height of 16, and so has area $4 \cdot 16 = 64$. Thus, the area of the region lies between 32 and 64.

(b) Since the region is symmetrical about the y-axis, we can find the area of half of the region and multiply by 2. An antiderivative for $16 - x^4$ is $16x - \frac{1}{5}x^5$.

$$\begin{aligned}2\int_0^2 (16 - x^4)\, dx &= 2\left(16(2) - \frac{1}{5} \cdot 2^5 - \left(16(0) - \frac{1}{5} \cdot 0^5\right)\right) \\ &= 2\left(32 - \frac{32}{5}\right) \\ &= 2\left(\frac{160}{5} - \frac{32}{5}\right) \\ &= \frac{256}{5} \\ &= 51.2.\end{aligned}$$

(c) We observe that $32 < 51.2 < 64$, so the area lies in the appropriate range.

3. (a) We can just count squares, or divide the region into triangles and trapezoids. We have an area of 2 below the x-axis, and an area of $\frac{17}{2}$ above the y-axis. Subtracting, we have an area of

$$\frac{17}{2} - 2 = \frac{17}{2} - \frac{4}{2} = \frac{13}{2}.$$

(b) We write the area as the sum of two integrals:

$$\int_{-4}^{1} (x + 2)\, dx + \int_1^3 (4 - x)\, dx.$$

For the first, we have an antiderivative of $\frac{1}{2}x^2 + 2x$, so that

$$\begin{aligned}\int_{-4}^{1}(x+2)\,dx &= \left(\frac{1}{2}\cdot 1^2 + 2\cdot 1 - \left(\frac{1}{2}(-4)^2 + 2(-4)\right)\right)\\ &= \frac{1}{2} + 2 - 8 + 8\\ &= \frac{5}{2}.\end{aligned}$$

For the second integral, we have an antiderivative of $4x - \frac{1}{2}x^2$, so that

$$\begin{aligned}\int_{1}^{3}(4-x)\,dx &= \left(4\cdot 3 - \frac{1}{2}\cdot 3^2 - \left(4\cdot 1 - \frac{1}{2}\cdot 1^2\right)\right)\\ &= 12 - \frac{9}{2} - 4 + \frac{1}{2}\\ &= 4.\end{aligned}$$

Adding these two areas, we get

$$\frac{5}{2} + \frac{8}{2} = \frac{13}{2}.$$

4. First, we see where the curves intersect.

$$\begin{aligned}x^2 - 4 &= 2 - x\\ x^2 + x - 6 &= 0\\ (x+3)(x-2) &= 0\\ x &= -3, 2\end{aligned}$$

We see that the line lies above the parabola, and so we evaluate

$$\int_{-3}^{2}(2 - x - (x^2 - 4))\,dx = \int_{-3}^{2}(6 - x - x^2)\,dx.$$

An antiderivative for $6 - x - x^2$ is

$$6x - \frac{1}{2}x^2 - \frac{1}{3}x^3,$$

and so the area is

$$\begin{aligned}6(2) - \frac{1}{2}(2)^2 - \frac{1}{3}(2)^3 - \left(6(-3) - \frac{1}{2}(-3)^2 - \frac{1}{3}(-3)^3\right) &= 12 - 2 - \frac{8}{3} - \left(-18 - \frac{9}{2} + 9\right)\\ &= \frac{125}{6}.\end{aligned}$$

NOTE: It would be OK to plug this in your calculator instead of work it out exactly. This is about 21, which is reasonable doing a quick square count on desmos.

5. First, we see where the curves intersect.

$$\begin{aligned} x^3 &= 4x \\ x^3 - 4x &= 0 \\ x(x^2-4) &= 0 \\ x(x+2)(x-2) &= 0 \\ x &= 0, -2, 2 \end{aligned}$$

Note that the line is on top. Since we are only looking at the area in Quadrant I, we need to evaluate

$$\int_0^2 (4x - x^3)\,dx.$$

An antiderivative for $4x - x^3$ is

$$2x^2 - \frac{1}{4}x^4$$

and so the area is

$$\begin{aligned} 2(2^2) - \frac{1}{4}(2^4) - \left(2(0)^2 - \frac{1}{4}(0)^4\right) &= 8 - 4 - 0 \\ &= 4 \end{aligned}$$

This is reasonable doing a quick square count on desmos.

9.4 The Area Function

Closely related to areas are **area functions.** Let's begin with an example. We see below in Figure 9.18 part of the graph of $f(x) = \dfrac{1}{x}$.

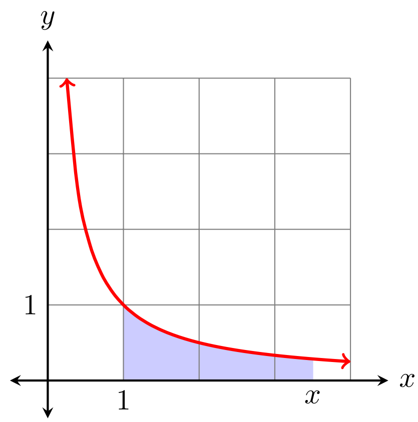


Figure 9.18: An area function.

First you need a starting point, which is $x_0 = 1$ in this case. You can use any value in the domain of the function. Often $x_0 = 0$ is easy to use, but our function is not defined there. Then you need a function, which is $f(x) = \dfrac{1}{x}$ in our case. With these you can define the **area function**

$$A(x) = \int_{x_0}^{x} f(u)\,du,$$

which in our case is

$$A(x) = \int_{1}^{x} \frac{1}{u}\,du.$$

What this means is that $A(x)$ gives you the area under a curve $f(x)$ and above the x-axis, measured starting from the point $x = x_0$. We use the variable u since it would be confusing to write

$$A(x) = \int_{x_0}^{x} f(x)\,dx,$$

because on one hand we're using x to represent a number on the x-axis, and on the other hand, we're using x as a variable in a function.

Let's take a moment to see why we can do this. Let's compute $\displaystyle\int_0^2 2u\,du$. Remember, the "$du$" means we're using the variable u in our functions. Then u^2 is an antiderivative for $2u$, and so

$$\int_0^2 2u\,du = 2^2 - 0^2 = 4.$$

But if we used a different variable, say "x," then an antiderivative of $2x$ would be x^2, and so

$$\int_0^2 2x\,dx = 2^2 - 0^2 = 4.$$

We get the same answer either way, since we're always plugging back into a function and evaluating. However, notice that

$$\int 2u\,du = u^2 + C, \qquad \int 2x\,dx = x^2 + C.$$

So when evaluating an *indefinite* integral, your answer will be in terms of whatever variable you're using. But when evaluating a definite integral, as we just saw, you'll always get the same answer no matter what variable you use.

So using "u" in

$$A(x) = \int_1^x \frac{1}{u}\,du$$

won't affect our answer. So let's work this out, remembering that $\ln u$ is an antiderivative of $1/u$.

$$\begin{aligned} A(x) &= \int_1^x \frac{1}{u}\,du \\ &= \ln x - \ln 1 \\ &= \ln x. \end{aligned}$$

The important observation here is that

$$A'(x) = \frac{1}{x} = f(x).$$

Notice that since $a = 1$ and $b = x$, our final answer is in terms of x.

But what if we used a *different* x_0 instead? Like $x_0 = 2$? Let's see.

$$\begin{aligned} A(x) &= \int_2^x \frac{1}{u}, du \\ &= \ln x - \ln 2. \end{aligned}$$

So we get a slightly different area function, since we're starting to measure area from a different place. But we *still* have

$$A'(x) = \frac{1}{x} = f(x).$$

We summarize this as follows.

Fundamental Theorem of Calculus, Part II

Let $f(x)$ be a continuous function, and let x_0 be a point in the domain of $f(x)$. If the area function $A(x)$ is defined by

$$A(x) = \int_{x_0}^x f(u)\,du,$$

then

$$A'(x) = f(x).$$

Again, we emphasize that $A(x)$ will depend upon what you choose for x_0, but whatever you choose, you will *always* get $A'(x) = f(x)$. We do need to assume that $f(x)$ is continuous, but this will be the case with all of our examples. We only need to include it for mathematical correctness, and you will notice this assumption if you look at other resources.

Often, the Fundamental Theorem of Calculus is stated without saying anything about area functions. Instead, it is stated as:

$$\frac{d}{dx}\int_{x_0}^{x} f(u)\,du = f(x).$$

This is equivalent to the way it is stated above, but you should be aware if you look at other resources.

Combined with the Fundamental Theorem of Calculus, Part I, we have essentially this. If you have a function $f(x)$, you antidifferentiate to get the corresponding area function. And if you are given an area function $A(x)$, you take the derivative to see what function it corresponds to. This is another way to say that differentiation and antidifferentiation are inverse processes.

Another way to say it is this. Suppose I give you a function, $f(x)$. I ask you two questions. First, find an antiderivative for $f(x)$. Next, write a function which calculates the area under the graph of $f(x)$ from a given starting point. Is there any reason to think that these two questions have the same answer? This is why the Fundamental Theorem of Calculus is so important.

It is also important to point out that we didn't actually *prove* the Fundamental Theorem of Calculus, but rather saw why it should be true through examples. That's good enough for our purposes. It is more important to understand what it means than how to prove it.

Example 1

We'll create an area function for $f(x) = x^2$ and $x_0 = -1$. The graph of $f(x) = x^2$ is shown at the left of Figure 9.19.

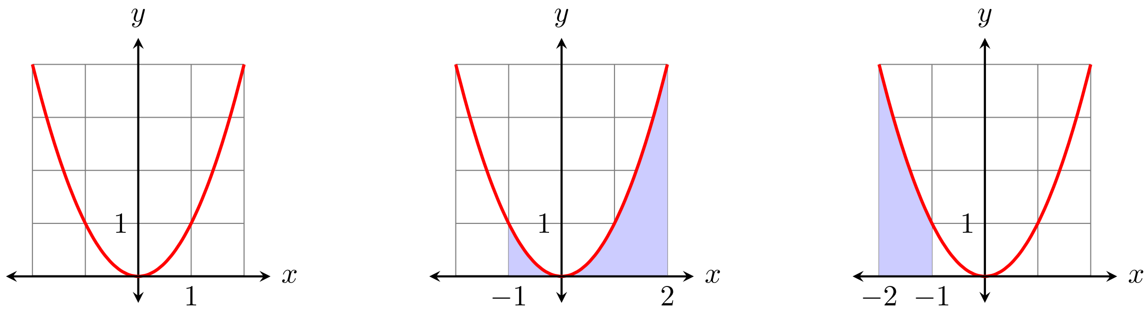


Figure 9.19: Areas bounded by $f(x) = x^2$.

$$\begin{aligned} A(x) &= \int_{-1}^{x} f(u)\,du \\ &= \int_{-1}^{x} u^2\,du \\ &= \frac{1}{3}u^3\bigg|_{-1}^{x} && \text{New notation for evaluating.} \\ &= \frac{1}{3}(x^3 - (-1)^3) \\ &= \frac{1}{3}(x^3 + 1). \end{aligned}$$

Note the new notation for evaluating. What is this notation good for? Suppose we wanted to find the area underneath $f(x) = x^2$ but above the x-axis on the interval $[-1, 2]$. On the one hand, this is just

$$A(2) = \frac{1}{3}(2^3 + 1) = 3.$$

Area functions are useful when you are evaluating many different areas. But if you just have to evaluate *one* area, which is often the case, this new notation lets you skip the area function altogether. You can just write

$$\begin{aligned} \int_{-1}^{2} f(x)\,dx &= \int_{-1}^{2} x^2\,dx \\ &= \frac{1}{3}x^3\bigg|_{-1}^{2} && \text{New notation for evaluating.} \\ &= \frac{1}{3}(2^3 - (-1)^3) \\ &= 3. \end{aligned}$$

Note that because our limits are *numbers,* we can use x as the variable. This is usually how it's done, but you can continue to use u if you want. Again, you need to be familiar with the standard notation in looking at other resources. If you need to find an *area function,* you'll need to use both x and u. But if you're just looking for an area, you can just use x. You should also be aware that many books use t instead of u, but we use t for time only.

Now let's look at the area underneath the parabola and above the x-axis on the interval $[-1,-2]$. So if we use the area function, we get that the area is

$$A(-2) = \frac{1}{3}((-2)^3 + 1) = -\frac{7}{3}.$$

But our area is *above* the x-axis, so how can it be negative? This is because we are measuring our area starting at $x = -1$, and have to go along the *negative* x-direction to get the area. Notice that

$$A(-2) = \int_{-1}^{-2} f(u)\, du.$$

Now let's evaluate (using our new notation) the area under the parabola and above the x-axis using $a = -2$ and $b = -1$:

$$\begin{aligned}
\int_{-2}^{-1} f(x)\, dx &= \int_{-2}^{-1} x^2\, dx \\
&= \frac{1}{3}x^3 \bigg|_{-2}^{-1} \\
&= \frac{1}{3}((-1)^3 - (-2)^3) \\
&= \frac{1}{3}(-1 + 8) \\
&= \frac{7}{3}.
\end{aligned}$$

This may look at little odd, but it is necessary. In geometry, areas were *always* positive. But in calculus, they can be negative. Because the Fundamental Theorem of Calculus is the central result relating differentiation/finding slopes to antidifferentiation/finding areas, it is *only* true if we introduce the concept of a negative area. So whether an area is positive or negative depends on two things: (1) whether the region is above or below the x-axis, or whether the region is traversed from the right or from the left. We see examples which summarize this in Figure 9.20.

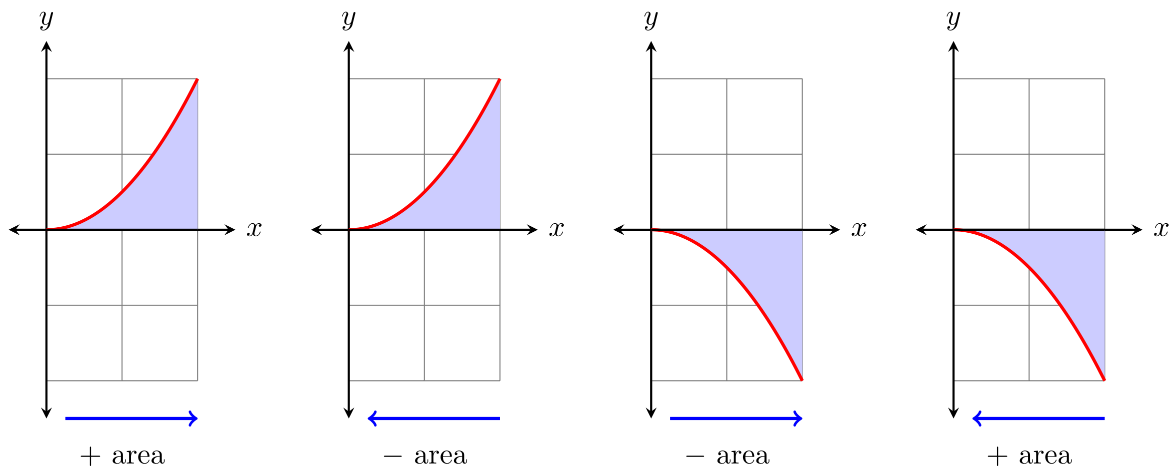


Figure 9.20: Above/below x-axis, traversing right/left.

Our example also showed that

$$\int_{-2}^{-1} f(x)\,dx = \frac{7}{3} = -\left(-\frac{7}{3}\right) = -\int_{-1}^{-2} f(x)\,dx.$$

This can be written in general; when a and b are in the domain of $f(x)$, then

$$\int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx.$$

This is because whether the area is above or below the x-axis, one integral measures the area from left to right, and the other measures from right to left. So they must be opposites of each other.

Example 2

Write an area function $A(x)$ for $f(x) = \sin(x)$ with $x_0 = 0$.

1. Verify that $A'(x) = f(x)$.
2. Find all the values of x where the area function is equal to 0.
3. Explain, using the graph of $\sin(x)$, why this makes sense geometrically.

Solution

To find the area function, calculate

$$\begin{aligned}\int_0^x \sin(u)\,du &= -\cos(u)\Big|_0^x \\ &= -\cos(x) - (-\cos(0)) \\ &= 1 - \cos(x)\end{aligned}$$

1. $$\frac{d}{dx}(1 - \cos(x)) = 0 - (-\sin(x)) = \sin(x).$$
2. Solving $A(x) = 0$ is the same as solving $\cos(x) = 1$. This occurs at all multiples of 2π: 0, 2π, 4π, etc., as well as -2π, -4π, etc.
3. Look at the graph of $\sin(x)$ on desmos. You will notice that starting at $x_0 = 0$ and going in either direction, every time you hit a multiple of 2π, you'll see that the positive areas *exactly* cancel out with the negative areas. Thus, the cumulative area determined by $A(x)$ must be 0.

Example 3

Suppose $f(x)$ is a function such that $\int_0^2 f(x)\,dx = -4$. What is $\int_2^0 f(x)\,dx$?

$\int_2^0 f(x)\,dx = -(-4) = 4$, since it still traces the area over the interval $[0,2]$, but in the *opposite* direction.

Example 4

Find $\dfrac{d}{dx}\int_2^x \arctan(u)\,du$.

From the Fundamental Theorem of Calculus, this is just $\arctan(x)$.

Example 5

Find $\dfrac{d}{dx}\int_x^3 e^{2u}\,dxu$

It is very important that the integral matches the Fundamental Theorem of Calculus *exactly.* The x must be in the upper limit, not the lower. So we need to switch first, as seen below.

$$\begin{aligned}\frac{d}{dx}\int_x^3 e^{2u}\,du &= -\frac{d}{dx}\int_3^x e^{2u}\,du \\ &= -e^{2x}.\end{aligned}$$

Example 6

Consider the function $f(x)$, graphed below.

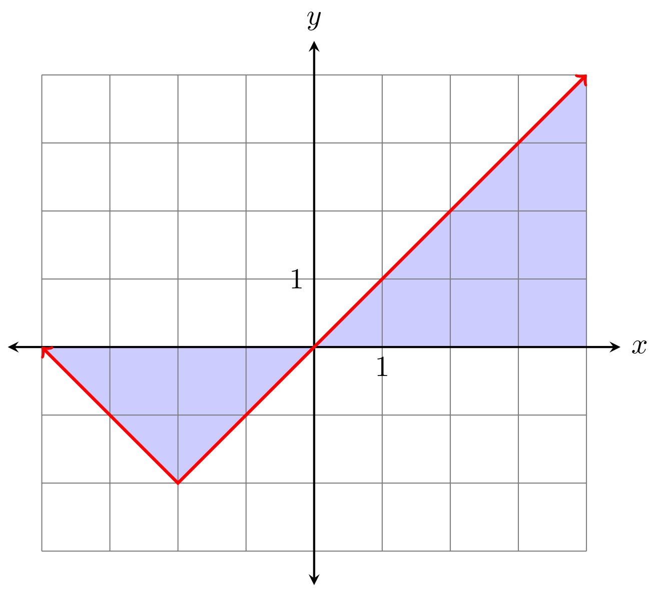


Figure 9.21: Evaluating areas.

Using simple geometry (no integrals needed), evaluate the following.

1. $\int_{-4}^{4} f(x)\,dx$.
2. $\int_{-2}^{2} f(x)\,dx$.
3. $\int_{4}^{0} f(x)\,dx$.
4. $\int_{3}^{-2} f(x)\,dx$.
5. $\int_{3}^{3} f(x)\,dx$.

Divide the areas into triangles and trapezoids. Remember that areas below the x-axis are negative, and traversing an interval from right to left changes the sign of the area.

Solutions

1. 4.
2. 0.
3. -8.
4. $-\frac{5}{2}$.
5. 0.

Homework

1. Find an area $A(x)$ function using $x_0 = 1$ and $f(x) = \dfrac{1}{x^4}$. Verify that $A'(x) = f(x)$.
2. Graph $f(x) = \arcsin(\sin(x))$ on desmos. Let $A(x)$ be the area function for $f(x)$ with $x_0 = 0$. Using geometry, find all values of x such that $A(x) = 0$.
3. Suppose $f(x)$ is a function such that $\int_2^6 f(x)\,dx = 10$. What is $\int_6^2 f(x)\,dx$?
4. Find $\dfrac{d}{dx}\displaystyle\int_3^x \ln(x^2+1)\,dx$.
5. Find $\dfrac{d}{dx}\displaystyle\int_x^7 \sin(3x-\pi)\,dx$.
6. Consider the function $f(x)$, graphed below.

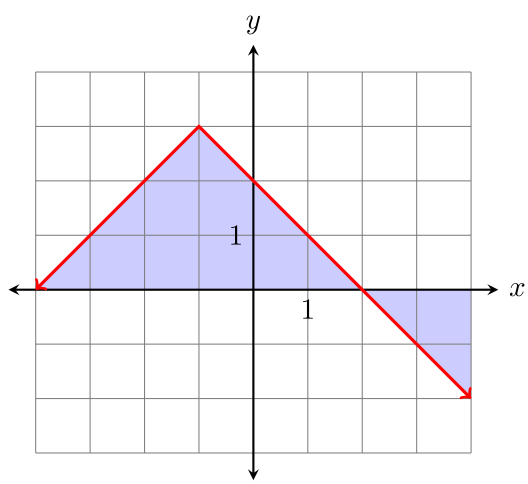


Figure 9.22: Evaluating areas.

Using simple geometry (no integrals needed), evaluate the following.

(a) $\displaystyle\int_{-4}^{4} f(x)\,dx$.

(b) $\displaystyle\int_{-2}^{4} f(x)\,dx$.

(c) $\displaystyle\int_{4}^{0} f(x)\,dx$.

(d) $\displaystyle\int_{2}^{-2} f(x)\,dx$.

(e) $\displaystyle\int_{1}^{1} f(x)\,dx$.

Solutions

1.

$$\begin{aligned} A(x) &= \int_1^x u^{-4}\,du \\ &= -\frac{1}{3}u^{-3}\Big|_1^x \\ &= -\frac{1}{3}x^{-3} + \frac{1}{3} \end{aligned}$$

Then

$$A'(x) = -\frac{1}{3}(-3x^{-4}) = \frac{1}{x^4}.$$

2. This problem is very similar to Example 2. Using the same logic, we have that $A(x) = 0$ when x is a multiple of 2π, including negative multiples as well.

3.

$$\int_6^2 f(x)\,dx = -\int_2^6 f(x)\,dx = -10.$$

4. Using the Fundamental Theorem of Calculus, this is just $\ln(x^2 + 1)$.

5. Using the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_x^7 \sin(3x - \pi)\,dx = -\frac{d}{dx}\int_7^x \sin(3x - \pi) = -\sin(3x - \pi).$$

6. (a) 7.
   (b) 5.
   (c) 0.
   (d) −7.
   (e) 0.

9.5 The Inverse Chain Rule

There are many techniques for finding antiderivatives. The first one is called using **substitution,** which we'll call the Inverse Chain Rule.

Let's review the Chain Rule by looking at $\frac{d}{dx}\sin(x^2)$. We use $f(x) = \sin(x)$ and $g(x) = x^2$, so that $f'(x) = \cos(x)$ and $g'(x) = 2x$. So

$$\begin{aligned}\frac{d}{dx}\sin(x^2) &= f'(g(x))g'(x)\\ &= \cos(g(x)) \cdot 2x\\ &= 2x\cos(x^2).\end{aligned}$$

Using antiderivative notation, we would write

$$\int 2x\cos(x^2)\,dx = \sin(x^2) + C.$$

We can generalize to any function composition by writing

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + C. \tag{9.1}$$

How would we use this if we were just given

$$\int 2x\cos(x^2)\,dx \ ? \tag{9.2}$$

We look at the **integrand,** which is the function to be integrated. Thus, $f'(g(x))g'(x)$ is the integrand in (9.1), and $2x\cos(x^2)$ is the integrand in (9.2).

Notice in (9.1) that you see a $g(x)$ and a $g'(x)$ in the integrand. That's our starting point. Looking at $2x\cos(x^2)$, can you see a $g(x)$ and a $g'(x)$? Yes, you've got x^2 and $2x$, and $2x$ is the derivative of x^2. So we know that $g(x) = x^2$.

How do we use this information? We make what is called a **substitution,** which is just $g(x)$. And just like with the Fundamental Theorem of Calculus, we need a different letter, which is usually u. In other words, we substitute $u = x^2$. You can see why we don't use x again, since writing $x = x^2$ is very confusing.

So $u = x^2$ and $\frac{du}{dx} = 2x$. Let's rewrite using this substitution.

$$\begin{aligned}\int 2x\cos(x^2)\,dx &= \int \cos(x^2)\cdot 2x\,dx\\ &= \int \cos(u)\cdot\frac{du}{dx}\cdot dx\\ &= \int \cos(u)\,du.\end{aligned}$$

It might look a little odd to cancel out the dx's. But this is an example of why the notation $\frac{du}{dx}$ is sometimes used instead of $u'(x)$. It makes the substitution process much easier.

Now the integral has become a lot simpler. In fact, this is one of the basic antiderivatives. So

$$\begin{aligned} \int \cos(u)\,du &= \sin(u) + C \\ &= \sin(x^2) + C. \qquad \text{substituting back, since } u = x^2 \end{aligned}$$

Example 1

Find $\displaystyle\int (1-2x)e^{x-x^2}\,dx$.

We first look at the integrand and see if can spot a $g(x)$ and $g'(x)$. Yes, $g(x) = x - x^2$ works, so we use the substitution $u = x - x^2$, so $\dfrac{du}{dx} = 1 - 2x$. Now rewrite.

$$\begin{aligned}
\int (1-2x)e^{x-x^2}\,dx &= \int e^{x-x^2}(1-2x)\,dx \\
&= \int e^u \cdot \frac{du}{dx} \cdot dx \\
&= \int e^u\,du \\
&= e^u + C \\
&= e^{x-x^2} + C.
\end{aligned}$$

You should notice that done correctly, substituting will eliminate *all* the x's, so the only variable will be u. If this does not happen, then you need to try another substitution.

There is another way to write out the substitution, which you will find in most other resources. It's the same algebra, though done in a slightly different order.

We start the same way: substitute $u = x - x^2$, so that $\dfrac{du}{dx} = 1 - 2x$. Now solve for du.

$$\begin{aligned}
\frac{du}{dx} &= 1 - 2x \\
\frac{du}{dx} \cdot dx &= (1-2x)\,dx \\
du &= (1-2x)\,dx
\end{aligned}$$

So when we rewrite, we get

$$\begin{aligned}
\int (1-2x)e^{x-x^2}\,dx &= \int e^{x-x^2}(1-2x)\,dx \\
&= \int e^u\,du \\
&= e^u + C \\
&= e^{x-x^2} + C.
\end{aligned}$$

Basically, we canceled out the dx's first by solving for du. We'll stick to this method, as it is more commonly used.

Example 2

Find $\int \sin(3x)\,dx$.

At first glance, it looks like we can't find both $g(x)$ and $g'(x)$. We might think $g(x) = 3x$ would work, but $g'(x) = 3$, and we don't see a 3.

But since 3 is a constant, we can multiply and divide by 3 as follows.

$$\begin{aligned}\int \sin(3x)\,dx &= \int \frac{1}{3}\cdot 3\sin(3x)\,dx \\ &= \frac{1}{3}\int 3\sin(3x)\,dx.\end{aligned}$$

Remember, we can factor constants out of derivatives and integrals, so the last step is legitimate. Now we have $g'(x)$, so we can make the substitution $u = 3x$, so that $\frac{du}{dx} = 3$ and $du = 3\,dx$. Rewriting, we get

$$\begin{aligned}\frac{1}{3}\int 3\sin(3x)\,dx &= \frac{1}{3}\int \sin(3x)\cdot 3\,dx \\ &= \frac{1}{3}\int \sin(u)\,du \\ &= \frac{1}{3}(-\cos(u)) + C \\ &= -\frac{1}{3}\cos(3x) + C\end{aligned}$$

IMPORTANT!!!!

This can **ONLY** be done because 3 is a number. You cannot do this otherwise.

$$\begin{aligned}\int \cos(x^2)\,dx &= \int \frac{1}{2x}\cdot 2x\cos(x^2)\,dx \\ &= \frac{1}{2x}\int 2x\cos(x^2)\,dx. \qquad \textbf{DON'T DO THIS!!!}\end{aligned}$$

So be careful. Only use this trick if your derivative is off by a **CONSTANT MULTIPLE.**

We summarize the steps below.

Inverse Chain Rule

To integrate $\displaystyle\int f'(g(x))g'(x)\,dx$:

1. Look for a $g(x)$ and $g'(x)$ pair in the integrand – $g'(x)$ can be off by a constant multiple;
2. If $g'(x)$ is off by a constant multiple, multiply and divide by this constant and factor out;
3. Substitute $u = g(x)$, and solve for $du = g'(x)\,dx$;
4. Rewrite the integral in terms of u; all x's should disappear;
5. Find the antiderivative with respect to u;
6. Substitute back to rewrite in terms of x only.

You'll only get better at substitution by practicing. The main trick is to spot $g(x)$ and $g'(x)$. Once you do this, just follow the steps one at a time. Be sure to have your table of antiderivatives handy.

Example 3

Find $\displaystyle\int x^3(x^4+2)^5\,dx$.

Let's go one step at a time.

1. We know that the derivative of x^4+2 is $4x^3$, which is good, since we're only off by a constant multiple of 4.

2. We now rewrite:
$$\int \frac{1}{4}\cdot 4\cdot x^3(x^4+2)^5\,dx = \frac{1}{4}\int 4x^3(x^4+2)^5\,dx.$$

3. We now substitute $u = x^4+2$ so that $\dfrac{du}{dx} = 4x^3$ and $du = 4x^3\,dx$.

4. Rewrite again:
$$\begin{aligned}\frac{1}{4}\int 4x^3(x^4+2)^5\,dx &= \frac{1}{4}\int (x^4+2)^5\cdot 4x^3\,dx\\ &= \frac{1}{4}\int u^5\,du\end{aligned}$$

5. Now take the antiderivative.
$$\begin{aligned}\frac{1}{4}\int u^5\,du &= \frac{1}{4}\left(\frac{1}{6}u^6\right)+C\\ &= \frac{1}{24}u^6+C.\end{aligned}$$

6. Finally, substitute back.
$$\begin{aligned}\int x^3(x^4+2)^5\,dx &= \frac{1}{24}u^6+C\\ &= \frac{1}{24}(x^4+2)^6+C.\end{aligned}$$

Example 4

Find $\int \frac{\ln x}{x}\,dx$.

1. Because the derivative of $\ln x$ is $\frac{1}{x}$, we choose $g(x) = \ln x$.
2. We've got it exactly, no need to adjust.
3. Now substitute $u = \ln x$, so that $du = \frac{1}{x}\,dx$.
4. Rewrite.
$$\int \ln x \cdot \frac{1}{x}\,dx = \int u\,du.$$
5. Taking the antiderivative, we have
$$\int u\,du = \frac{1}{2}u^2 + C.$$
6. Substituting back, we get
$$\begin{aligned}\int \frac{\ln x}{x}\,dx &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}(\ln x)^2 + C.\end{aligned}$$

Example 5

Find $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,dx$.

This problem is similar to Example 2. We notice that it looks pretty close to the derivative of $\arcsin(x)$, except for the factor of 4.

1. Let's see what happens if we make $g(x) = 2x$. Then $g'(x) = 2$. Remember, we can be off by a constant multiple.
2. So we can rewrite as
$$\frac{1}{2}\int \frac{2}{\sqrt{1-4x^2}}\,dx.$$
3. Now substitute $u = 2x$, so that $4x^2 = (2x)^2 = u^2$ and $du = 2\,dx$.
4. Rewriting again, we have
$$\begin{aligned}\frac{1}{2}\int \frac{2}{\sqrt{1-4x^2}}\,dx &= \frac{1}{2}\int \frac{1}{\sqrt{1-(2x)^2}}\cdot 2\,dx \\ &= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du\end{aligned}$$
Can you see why using $g(x) = 2x$ was a good idea? To get an $\arcsin(x)$ in our answer, the derivative has to match *exactly.* In a sense, the substitution $u = 2x$ allows us to get rid of the factor of 4. This technique is useful for the derivatives of inverse trigonometric functions.
5. Now we can take the antiderivative.
$$\frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{2}\arcsin(u) + C.$$
6. Substituting back, we get
$$\begin{aligned}\int \frac{1}{\sqrt{1-4x^2}}\,dx &= \frac{1}{2}\arcsin(u) + C \\ &= \frac{1}{2}\arcsin(2x) + C.\end{aligned}$$

Homework

1. Find $\int x^5(1-x^6)^3\,dx$.
2. Find $\int 3x^2\sin(x^3)\,dx$.
3. Find $\int e^{5x}\,dx$.
4. Find $\int \frac{1}{1+9x^2}\,dx$.
5. Find $\int \frac{(\ln x)^2}{3x}\,dx$.
6. Find $\int x\,3^{x^2}\,dx$.
7. Find $\int e^x\cos(e^x)\,dx$.
8. Find $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$.

Solutions

1. Use $u = 1 - x^6$, so that $\dfrac{du}{dx} = -6x^5$ and $du = -6x^5\,dx$. We're off by a factor of -6, so we compensate and then substitute.

$$\begin{aligned}\int x^5(1-x^6)^3\,dx &= \frac{1}{-6}\int -6x^5(1-x^6)^3\,dx\\ &= -\frac{1}{6}\int (1-x^6)^3(-6x^5\,dx)\\ &= -\frac{1}{6}\int u^3\,du\\ &= -\frac{1}{6}\cdot\frac{1}{4}u^4 + C\\ &= -\frac{1}{24}(1-x^6)^4 + C.\end{aligned}$$

2. Use $u = x^3$, so that $\dfrac{du}{dx} = 3x^2$ and $du = 3x^2\,dx$. It's an exact match, so no need to compensate.

$$\begin{aligned}\int 3x^2\sin(x^3)\,dx &= \int \sin(x^3)\cdot 3x^2\,dx\\ &= \int \sin(u)\,du\\ &= -\cos(u) + C\\ &= -\cos(x^3) + C.\end{aligned}$$

3. We use $u = 5x$ so that $\dfrac{du}{dx} = 5$ and $du = 5\,dx$. We're off by a factor of 5, we we'll need to compensate.

$$\begin{aligned}\int e^{5x}\,dx &= \frac{1}{5}\int 5e^{5x}\,dx\\ &= \frac{1}{5}\int e^{5x}\cdot 5\,dx\\ &= \frac{1}{5}\int e^u\,du\\ &= \frac{1}{5}e^u + C\\ &= \frac{1}{5}e^{5x} + C.\end{aligned}$$

4. This looks like an $\arctan(x)$ will be involved. We rewrite as $\displaystyle\int \frac{1}{1+(3x)^2}\,dx$, and so we use $u = 3x$. Then $\dfrac{du}{dx} = 3$ and $du = 3\,dx$. We're off by a factor of 3.

$$\begin{aligned}\int \frac{1}{1+9x^2}\,dx &= \frac{1}{3}\int 3\cdot\frac{1}{1+(3x)^2}\,dx\\ &= \frac{1}{3}\int \frac{1}{1+(3x)^2}\cdot 3\,dx\\ &= \frac{1}{3}\int \frac{1}{1+u^2}\,du\\ &= \frac{1}{3}\arctan(u)+C\\ &= \frac{1}{3}\arctan(3x)+C.\end{aligned}$$

5. We use $u = \ln x$, so that $\dfrac{du}{dx} = \dfrac{1}{x}$ and $du = \dfrac{1}{x}\,dx$. We don't need the 3, so we just factor it out.

$$\begin{aligned}\int \frac{(\ln x)^2}{3x}\,dx &= \frac{1}{3}\int (\ln x)^2\cdot\frac{1}{x}\,dx\\ &= \frac{1}{3}\int u^2\,du\\ &= \frac{1}{3}\cdot\frac{1}{3}u^3+C\\ &= \frac{1}{9}(\ln x)^3+C.\end{aligned}$$

6. We use $u = x^2$, so that $\dfrac{du}{dx} = 2x$ and $du = 2\,dx$. We're off by a factor of 2.

$$\begin{aligned}\int x\,3^{x^2}\,dx &= \frac{1}{2}\int 2x\,3^{x^2}\,dx\\ &= \frac{1}{2}\int 3^{x^2}\cdot 2x\,dx\\ &= \frac{1}{2}\int 3^u\,du\\ &= \frac{1}{2}\cdot\frac{3^u}{\ln 3}+C\\ &= \frac{3^{x^2}}{2\ln 3}+C.\end{aligned}$$

7. We use $u = e^x$, so that $\dfrac{du}{dx} = e^x$ and $du = e^x\,dx$. We've got an exact match.

$$\begin{aligned}\int e^x \cos(e^x)\,dx &= \int \cos(e^x)\cdot e^x\,dx \\ &= \int \cos(u)\,du \\ &= \sin(u) + C \\ &= \sin(e^x) + C.\end{aligned}$$

8. In the denominator, we see an expression which looks like an $\arcsin(x)$ is involved. Rerwriting as $\displaystyle\int \frac{e^x}{\sqrt{1-(e^x)^2}}\,dx$, we use the substitution $u = e^x$. Then $\dfrac{du}{dx} = e^x$ and $du = e^x\,dx$, which gives us an exact match.

$$\begin{aligned}\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx &= \int \frac{1}{\sqrt{1-(e^x)^2}}\cdot e^x\,dx \\ &= \int \frac{1}{\sqrt{1-u^2}}\,du \\ &= \arcsin(u) + C \\ &= \arcsin(e^x) + C.\end{aligned}$$