Some further ideas about rules for differentiation

We emphasize the deriviative as a linear approximation to a function at a point; in other words,

$$f(x+h) \approx f(x) + hf'(x).$$

Of course, the second-order approximation is just

$$f(x+h) \approx f(x) + hf'(x) + \frac{1}{2}h^2 f''(x).$$

(Does anyone know of a nice way to motivate this without involving series? I've found a way, but it's not particularly insightful – and not extremely rigorous, but I think it can be made so. Suppose

$$f(x+h) \approx f(x) + hf'(x) + h^2 g(x).$$

Taking derivatives with respect to h gives

$$f'(x+h) \approx f'(x) + 2hg(x),$$

so that

$$g(x) \approx \frac{1}{2}\frac{f'(x+h) - f'(x)}{h}.$$

Taking the limit as $h \to 0$ gives

$$g(x) \approx \frac{1}{2}f''(x).$$

A slicker way would be nice. Of course it is always possible to look numerically at

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - hf'(x)}{h^2}$$

for various values of x, and choose a function for which it is easy to guess $f''(x)/2$.)

This helps to make clear the idea of "first-order" approximation. Now using first order approximations for f and g results in

$$\begin{aligned} f(x+h)g(x+h) &\approx (f(x) + hf'(x))(g(x) + hg'(x)) \\ &= f(x)g(x) + h(f'(x)g(x) + f(x)g'(x)) + h^2 f'(x)g'(x). \end{aligned}$$

Thus,

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \approx f'(x)g(x) + f(x)g'(x) + hf'(x)g'(x).$$

Taking limits, we obtain the product rule.

I think this might be a way to motivate the product rule in a manner students can be guided to work out on their own. Using the limit definition requires the trick of adding and subtracting $f(x)g(x+h)$ (or $f(x+h)g(x)$), which would be good to show them *anyway,* and perhaps having them derive the result and have it at hand might be enought to suggest to them the trick.

It turns out it that this same idea can be applied to the quotient rule, which involves not only thinking about first-order approximations, but also geometric series. We first need to approximate

$$\frac{f(x+h)}{g(x+h)} \approx \frac{f(x)+hf'(x)}{g(x)+hg'(x)}.$$

Thus, we need a first order approximation to

$$\frac{1}{g(x)+hg'(x)}.$$

Here's where geometric series come in: we may write

$$\frac{1}{a+hb} = \frac{\frac{1}{a}}{1+h\frac{b}{a}} = \frac{1}{a}\left(1 - h\frac{b}{a} + h^2\frac{b^2}{a^2} - \cdots\right).$$

Since we are letting h go to 0, we can be sure that this geometric series converges. So to first order,

$$\frac{1}{a+hb} \approx \frac{1}{a} - h\frac{b}{a^2},$$

and applied to the question at hand we have

$$\frac{1}{g(x)+hg'(x)} \approx \frac{1}{g(x)} - h\frac{g'(x)}{g(x)^2}.$$

Multiplying both first-order approximations gives

$$\frac{f(x+h)}{g(x+h)} \approx (f(x)+hf'(x))\left(\frac{1}{g(x)} - h\frac{g'(x)}{g(x)^2}\right) = \frac{f(x)}{g(x)} + h\frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2} - h^2\frac{f'(x)g'(x)}{g(x)^2}.$$

Looking at the first-order term gives the quotient rule.

This method can also be applied to the product rule involving three functions. By writing

$$f(x+h)g(x+h)k(x+h) \approx (f(x)+hf'(x))(g(x)+hg'(x))(k(x)+hk'(x)),$$

we may look at just the h term to find the appropriate derivative formula:

$$f'(x)g(x)k(x) + f(x)g'(x)k(x) + f(x)g(x)k'(x).$$

Interestingly, the chain rule can also be proved using the same method. We simply need to approximate

$$f(g(x+h)) \approx f(g(x)+hg'(x)).$$

But for a given x, $hg'(x)$ goes to 0 with h, so we can use a linear approximation to f to get

$$f(g(x+h)) \approx f(g(x)+hg'(x)) \approx f(g(x)) + hg'(x)f'(g(x)),$$

which gives the chain rule.

What about derivatives of the trigonometric functions? Perhaps these can be done the same way. Let's look at the sine function:

$$\sin(x+h) = \sin x \cos h + \sin h \cos x.$$

Now a first-order approximation to $\cos h$ is 1, while a first-order approximation to $\sin h$ is h; this is the essential meaning of the statement that

$$\lim_{h\to 0} \frac{\sin h}{h} = 1.$$

Thus, to first-order,

$$\sin(x+h) \approx \sin x + h\cos x.$$

But this is just the statement that $\frac{d}{dx}\sin x = \cos x$. In looking carefully at the proof involving limits, the same conceptual ideas are used, but stated in a different way.

We may find the derivative of the tangent function in the same way. We begin with

$$\tan(x+h) = \frac{\tan x + \tan h}{1 - \tan x \tan h}.$$

Now since

$$\lim_{h\to 0} \frac{\tan h}{h} = 1,$$

h is a first-order approximation to $\tan h$. Thus

$$\tan(x+h) \approx \frac{\tan x + h}{1 - h\tan x}.$$

Now use a geometric series to get a first-order approximation to $1/(1 - h\tan x)$:

$$\frac{1}{1 - h\tan x} \approx 1 + h\tan x.$$

Thus,

$$\tan(x+h) \approx (\tan x + h)(1 + h\tan x) = \tan x + h(1+\tan^2 x) + h^2 \tan x \approx \tan x + h\sec^2 x.$$

Therefore the derivative of $\tan x$ is $\sec^2 x$.

The derivative of $\sec x$ is particularly easy using method. To first-order, we have

$$\sec(x+h) = \frac{1}{\cos(x+h)} = \frac{1}{\cos x\cos h - \sin x \sin h} \approx \frac{1}{\cos x - h\sin x}.$$

But our geometric series trick gives

$$\sec(x+h) \approx \frac{1}{\cos x}\left(1 + h\frac{\sin x}{\cos x}\right) = \sec x + h \sec x \tan x.$$

This gives $\sec x \tan x$ as the derivative of $\sec x$.

Of course these same ideas can be used to find the derivatives of the hyperbolic trigonometric functions. But interestingly, they can be used in an unusual way to find the derivative of e^x. By looking at the graphs of $\cosh x$ and $\sinh x$, we can see that 1 must be a first-order approximation to $\cosh h$ since the tangent is horizontal there, and by zooming graphically toward 0, we see that h is a first-order approximation to $\sinh x$. Thus,

$$\sinh(x+h) = \sinh x \cosh h + \cosh x \sinh h \approx \sinh x + h \cosh x,$$

so that $\cosh x$ is the derivative of $\sinh x$. Similarly, we can see that $\sinh x$ is the derivative of $\cosh x$. Then, since $e^x = \cosh x + \sinh x$, we have

$$\frac{d}{dx}e^x = \frac{d}{dx}\cosh x + \frac{d}{dx}\sinh x = \sinh x + \cosh x = e^x.$$

We may avoid the graphical observation that $\sinh h \approx h$ (we can do the same to find the derivative of e^x at 0) by writing (see the comments on hyperbolic functions)

$$e^x = \cosh x + \sinh x = \cos(ix) - i\sin(ix).$$

But it is easy to calculate that $\cos(ix) - i\sin(ix)$ is its own derivative, and so must be e^x.

This same way of thinking is essential in looking at derivatives of vector functions. Just look at the GRAND DEFINITION OF DIFFERENTIABILITY on p. 118 of Colley. As an example, given the function $f(\mathbf{v}) = \mathbf{v}\cdot\mathbf{v}$, we find its derivative by looking at

$$f(\mathbf{v}+\mathbf{h}) = (\mathbf{v}+\mathbf{h})\cdot(\mathbf{v}+\mathbf{h}) = \mathbf{v}\cdot\mathbf{v} + 2\mathbf{v}\cdot\mathbf{h} + \mathbf{h}\cdot\mathbf{h}.$$

Thus, a first-order approximation is

$$f(\mathbf{v}+\mathbf{h}) \approx f(\mathbf{v}) + 2\mathbf{h}\cdot\mathbf{v},$$

so that

$$\nabla_{\mathbf{v}} f(\mathbf{h}) = 2\mathbf{h}\cdot\mathbf{v}.$$

And again, as these students will be prime candidates for MVC, it would be advantageous for them to see this way of thinking before they are in MVC.

Perhaps more compellingly, proofs of the usual rules for differentiation are rather straightforward and more intuitive than proofs involving the manipulation of limits.

Here are a few more examples of the method, for your amusement.

Let's find the derivative of x^{-n}. We have

$$\frac{1}{(x+h)^n} \approx \left(\frac{1}{x}\left(1-\frac{h}{x}\right)\right)^n \approx \frac{1}{x^n}\left(1-\frac{nh}{x}\right) = \frac{1}{x^n} + h\frac{-n}{x^{n+1}},$$

so that $-nx^{-n-1}$ is the derivative of x^{-n}.

Now consider $\sqrt[n]{x}$; we must approximate $\sqrt[n]{x+h}$. Thus, we seek a such that

$$x + h \approx (\sqrt[n]{x} + ah)^n \approx x + n\sqrt[n]{x}^{n-1}ah.$$

Thus,

$$an\sqrt[n]{x}^{n-1} = 1,$$

giving

$$a = \frac{1}{n}x^{\frac{1-n}{n}} = \frac{1}{n}x^{\frac{1}{n}-1},$$

and so we have found the derivative of $\sqrt[n]{x}$.

We may also see one form of L'Hôpital's rule this way. Suppose we wish to find

$$\lim_{x\to a}\frac{f(x)}{g(x)},$$

and further that $f(a) = g(a) = 0$. Writing with first order approximations, we have

$$\frac{f(x)}{g(x)} \approx \frac{f(x) + hf'(x)}{g(x) + hg'(x)}.$$

Now substitute $x = a$; the h cancels out and we are left with

$$\frac{f'(a)}{g'(a)}.$$

Yes, this is incomplete, but perhaps it gives a motivation for why L'Hôpital's rule is valid.