Geometrical Transformations: Groups and Matrices

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Brief Description of IMSA

- Established in 1985;
- Funded by the State of Illinois;
- Grades 10–12;
- Residential school with 650 students, about half male, half female;
- Geared toward gifted students in mathematics and science;
- Strong programs in the humanities;
- Specific outreach to underserved students;
- Many summer programs for younger students;
- Professional development opportunities for teachers;
- Houses a Problem-Based Learning Network.

We can look at geometrical transformations from three perspectives

First, we may consider them as transformations which have an effect on geometric objects (such as polygons).

Second, we may look at the abstract structure of the transformations (using group theory and abstract algebra).

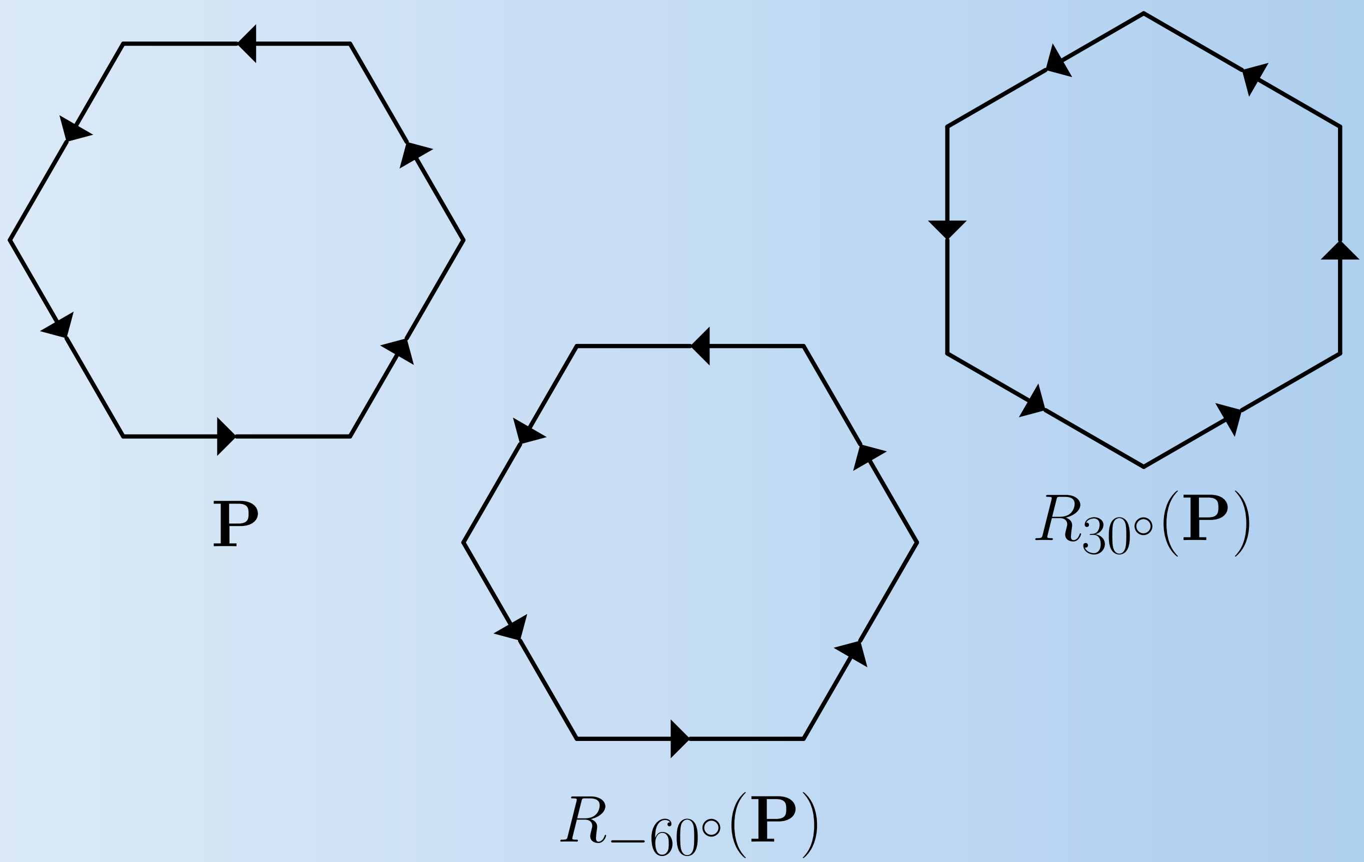
Third, we may consider them as algebraic objects (that is, matrices).

Different perspectives give different insights into the nature of geometric transformations.

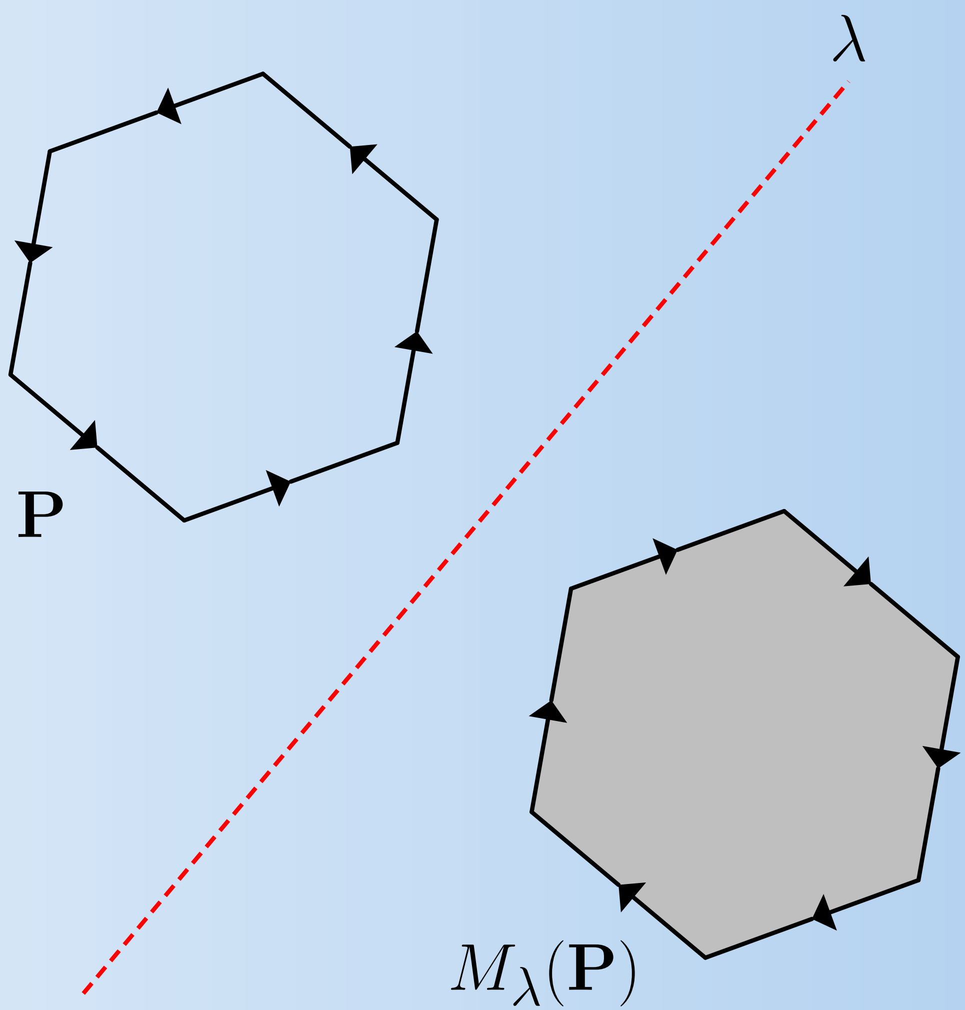
The purpose of this workshop is to enlarge your mathematical vocabulary, which will in turn enable you to consider geometric transformations from multiple perspectives.

This will help you as you consider how to communicate important mathematical ideas to students.

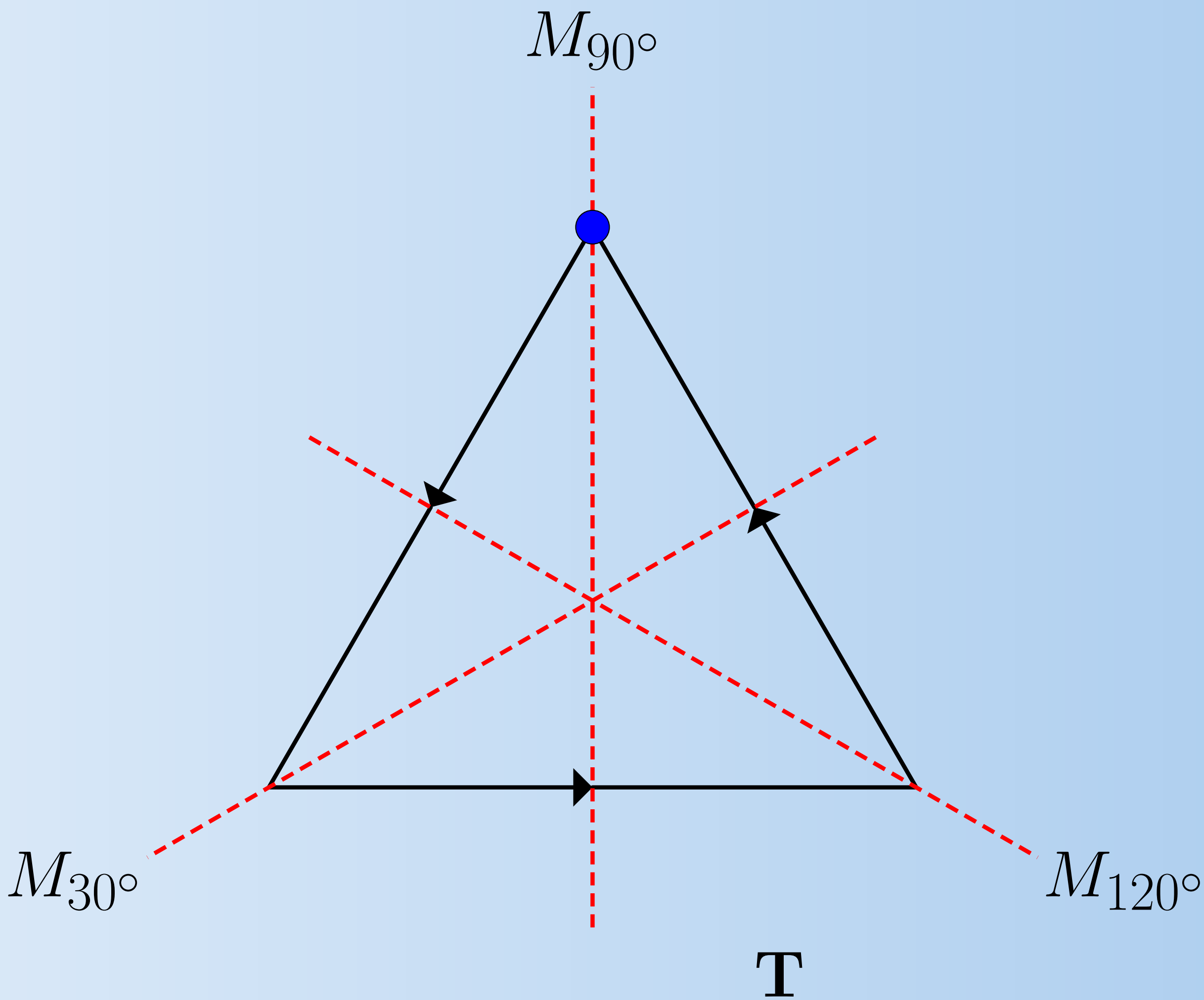
The *rotation* R_θ rotates a figure about the origin counter-clockwise through an angle θ. This means that if $\theta < 0$, the figure is rotated clockwise.



The *reflection* M_λ reflects a figure about the line λ. The figure is "turned over."

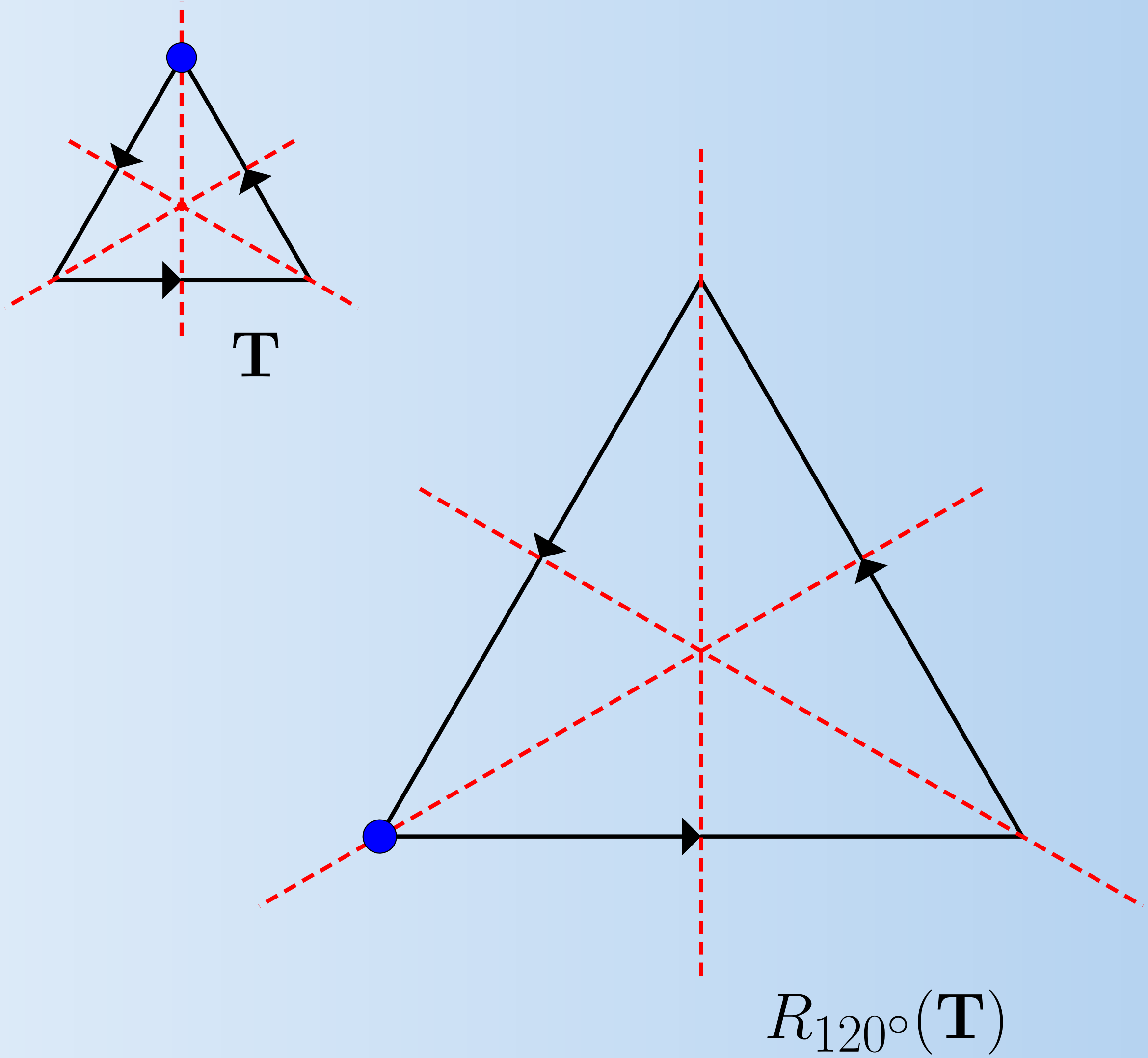


We begin by exploring the symmetries of the triangle.

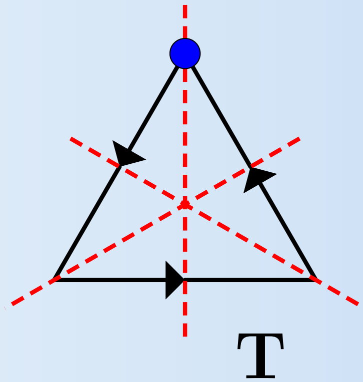


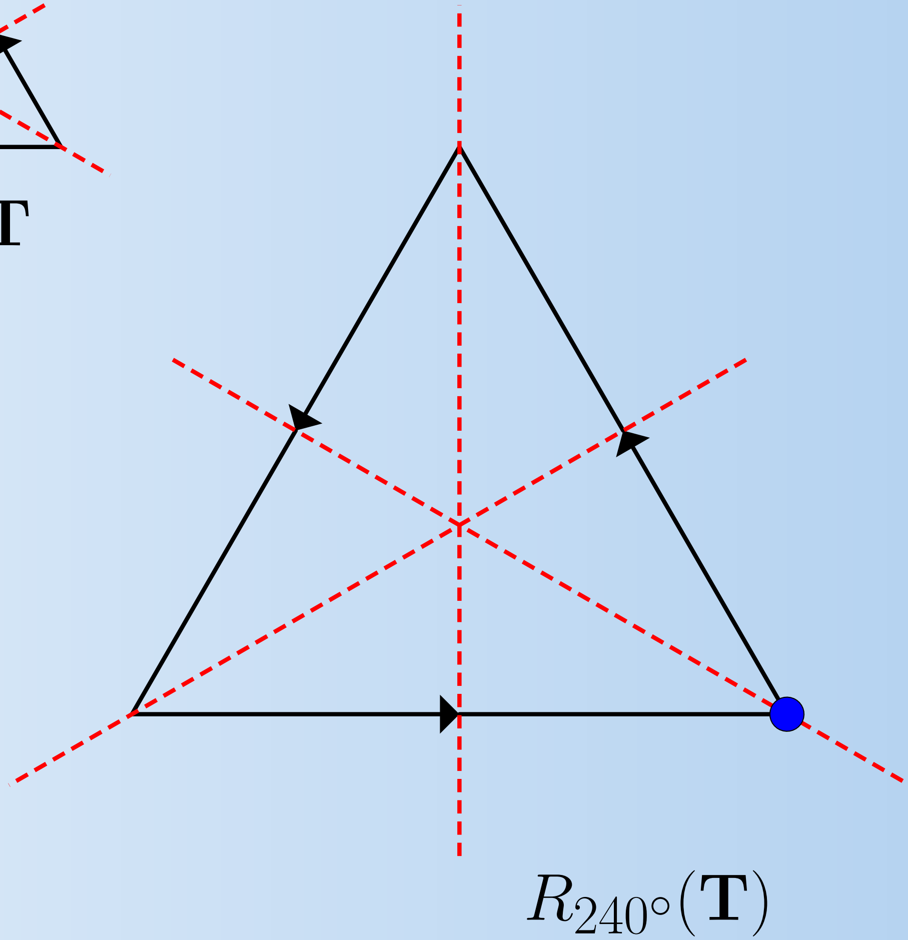
The Rotation R_{120°

R_{120° rotates the triangle 120° counterclockwise.



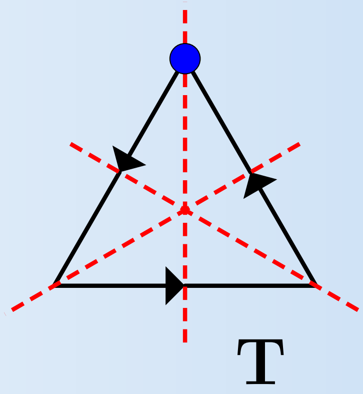
R_{240° rotates the triangle 240° counterclockwise.

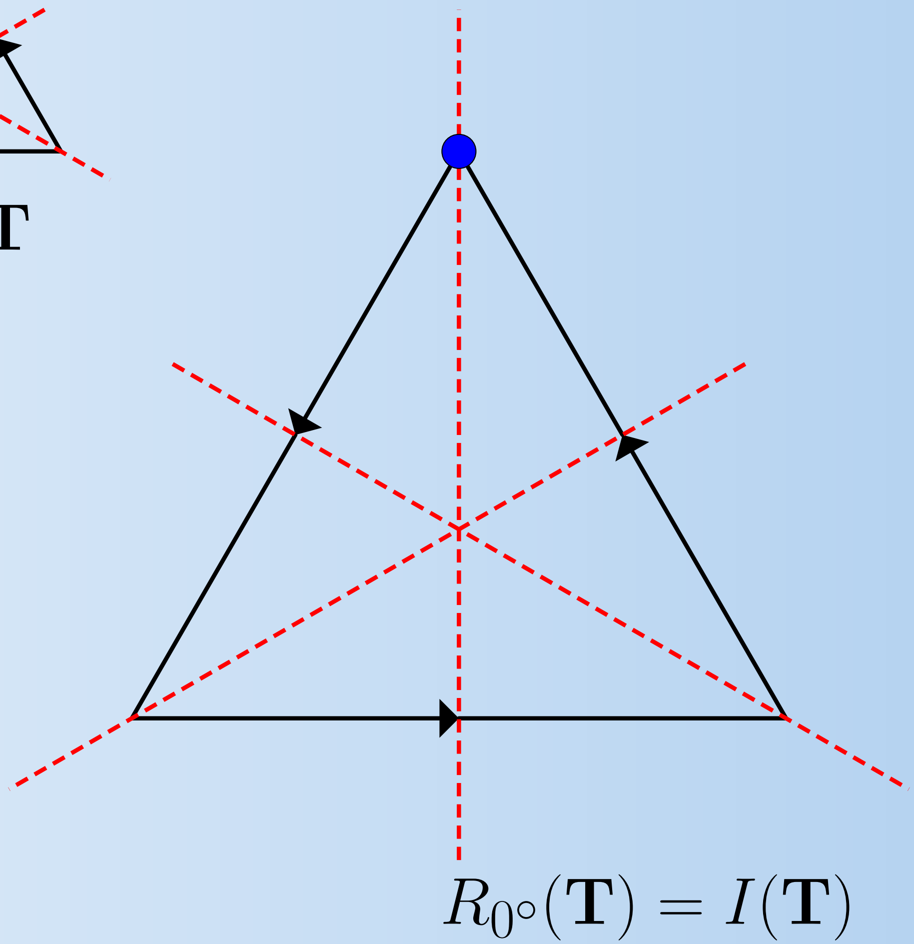




The Rotation R_{0°

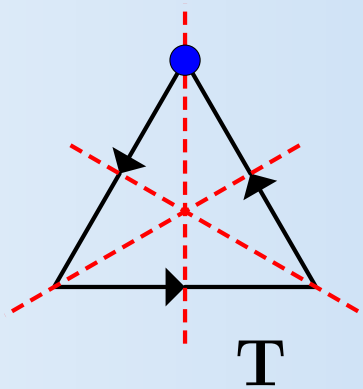
R_{0° is the *identity transformation.*

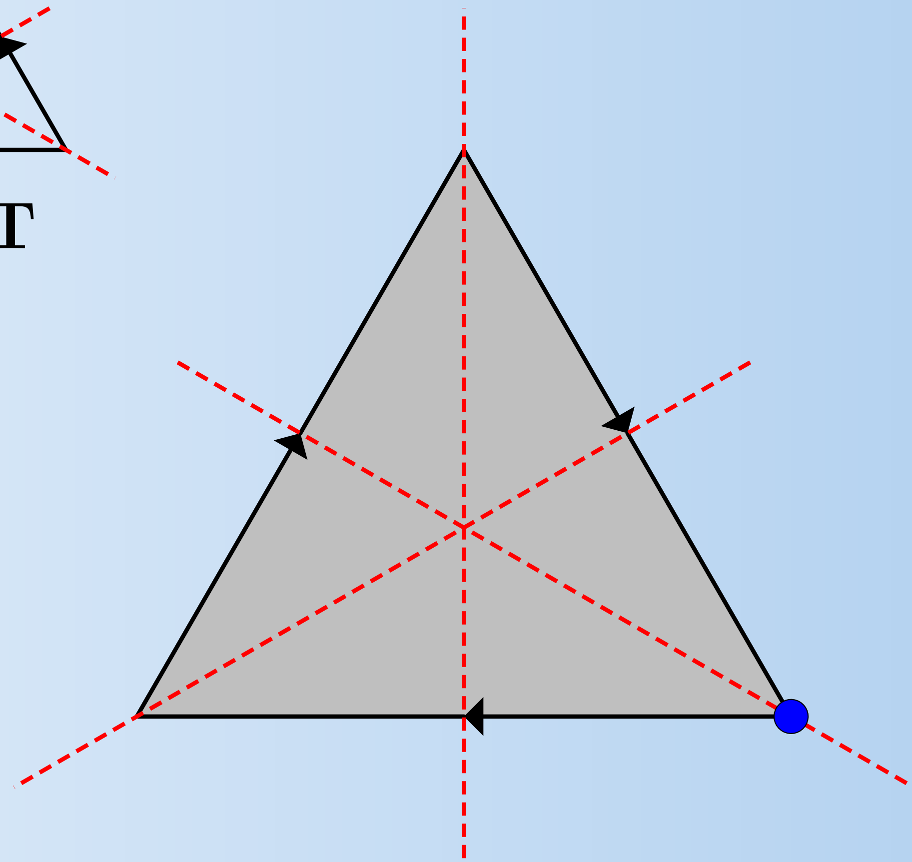




The Reflection M_{30°

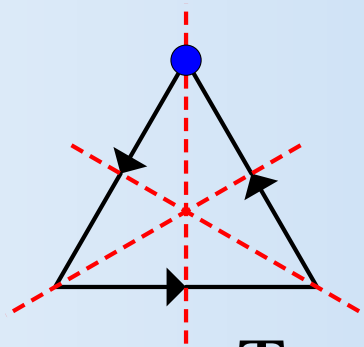
M_{30° reflects the triangle along a line through the origin making a 30° angle with the positive x-axis.





The Reflection M_{90°

M_{90° reflects the triangle along a line through the origin making a 90° angle with the positive x-axis.



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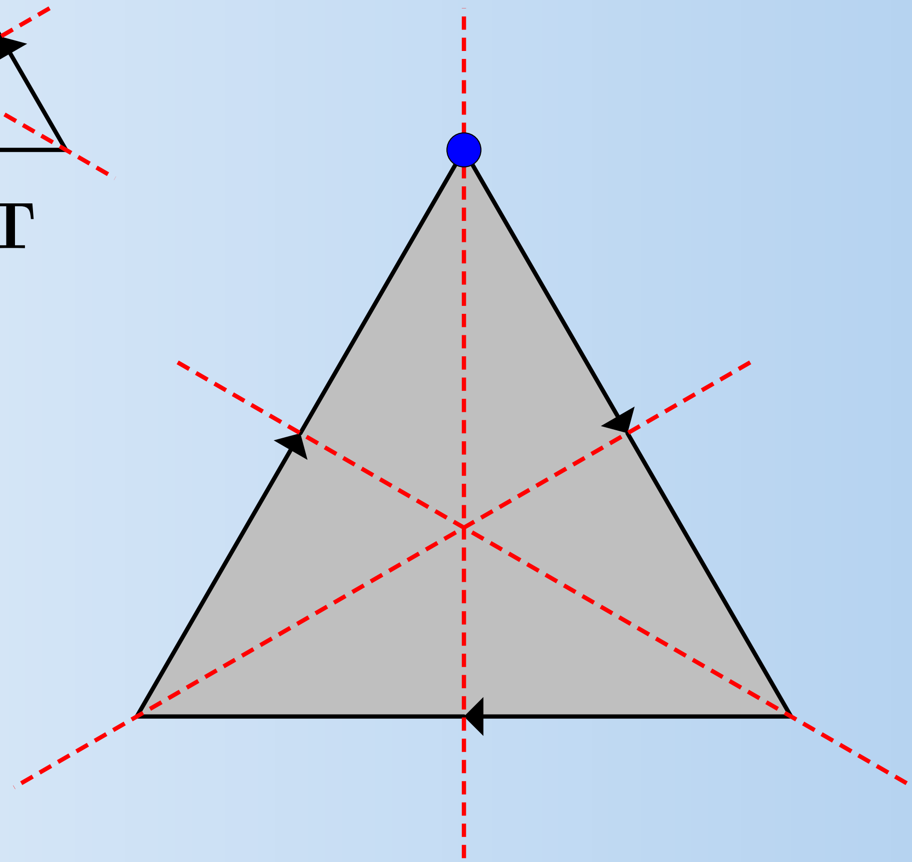
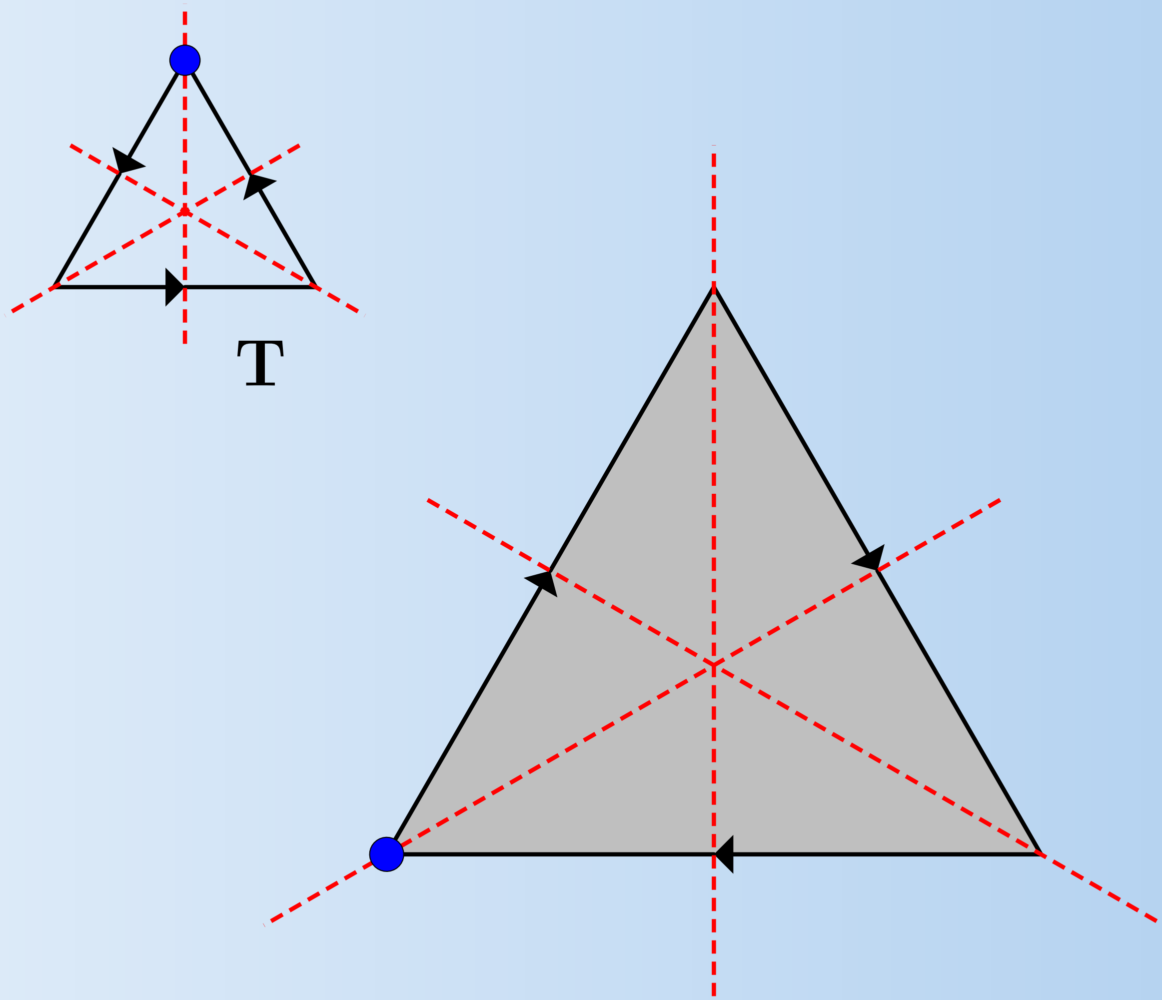
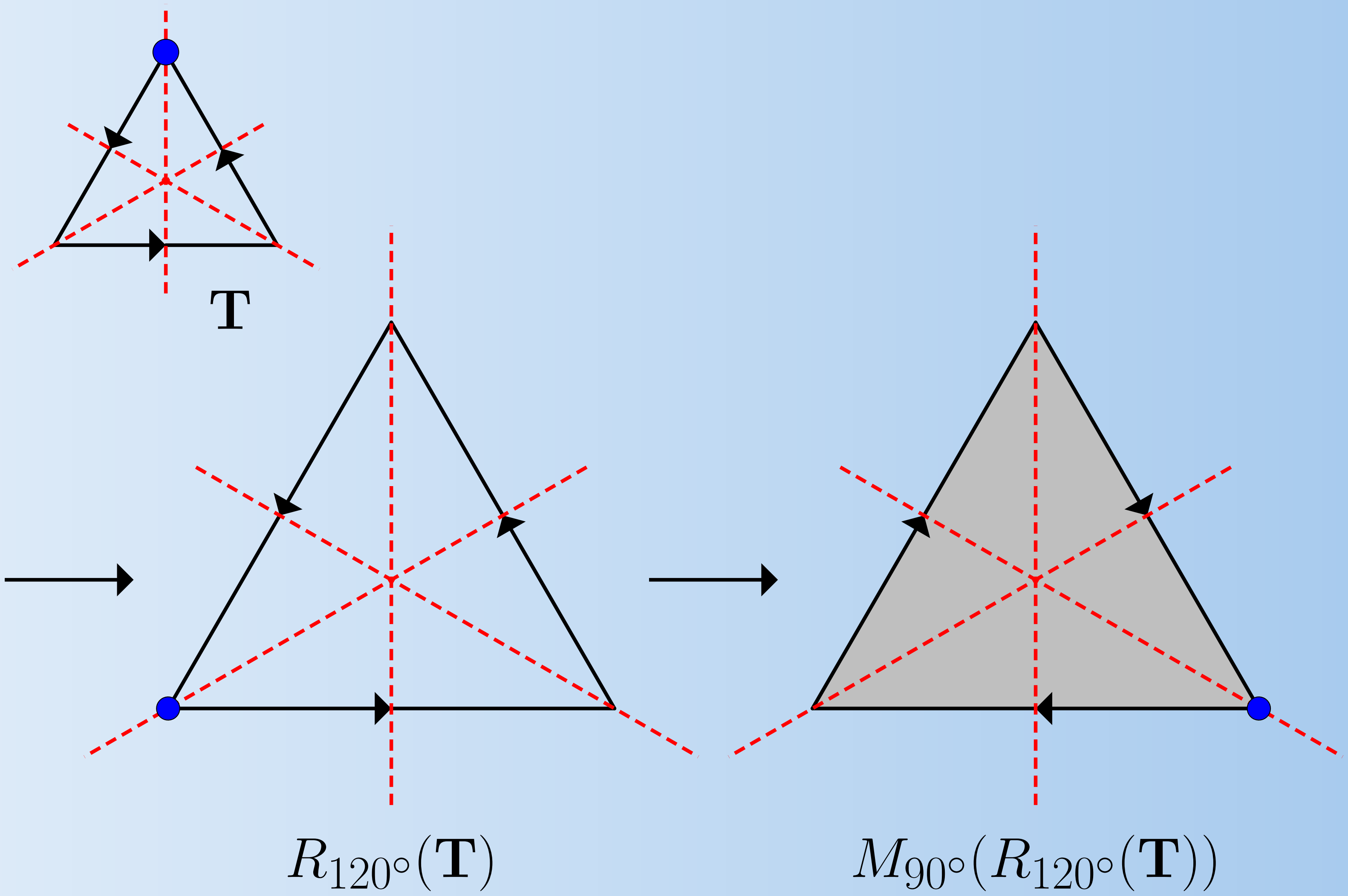


The Reflection M_{120°

M_{120° reflects the triangle along a line through the origin making a 120° angle with the positive x-axis.

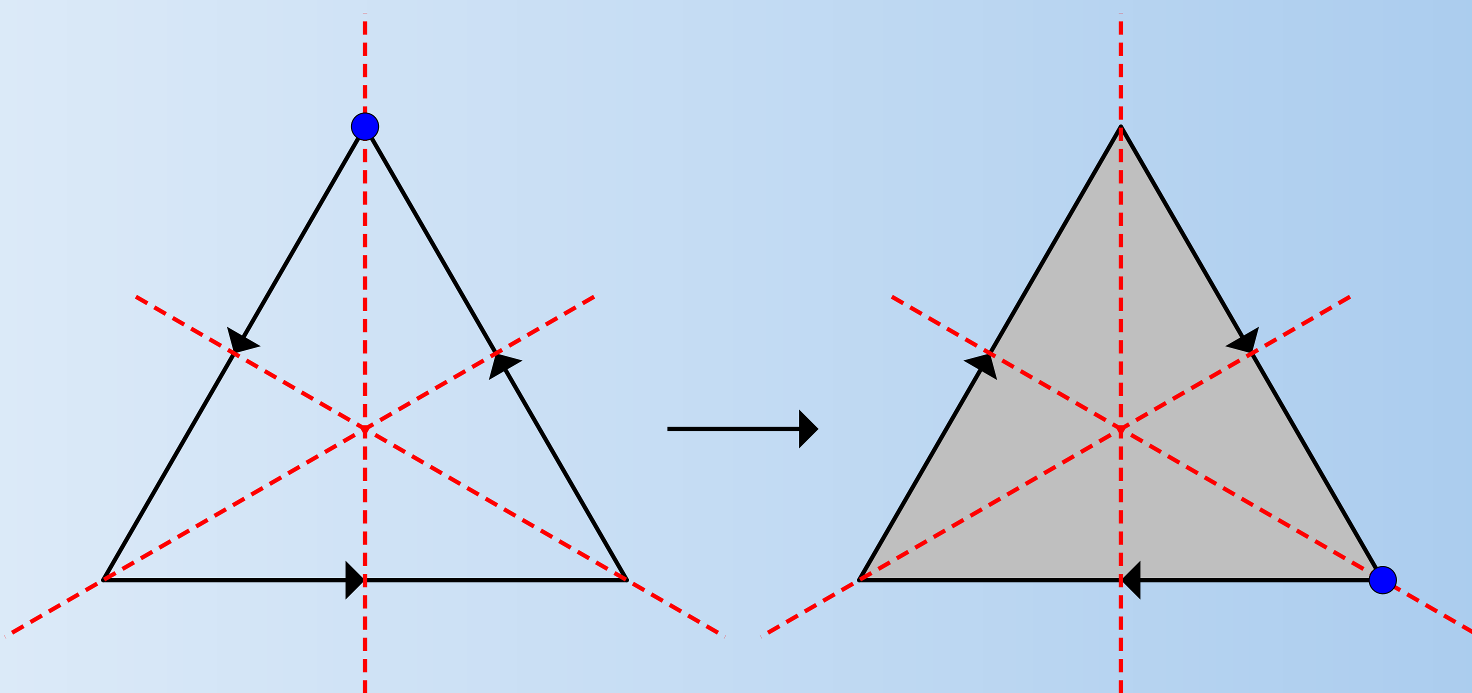


Suppose we first rotate the triangle using R_{120°, and then reflect it with M_{90°.



More Composition of Symmetries

We call this the *composition* of M_{90° and R_{120°, written $M_{90^\circ} \circ R_{120^\circ}$. Note the order!

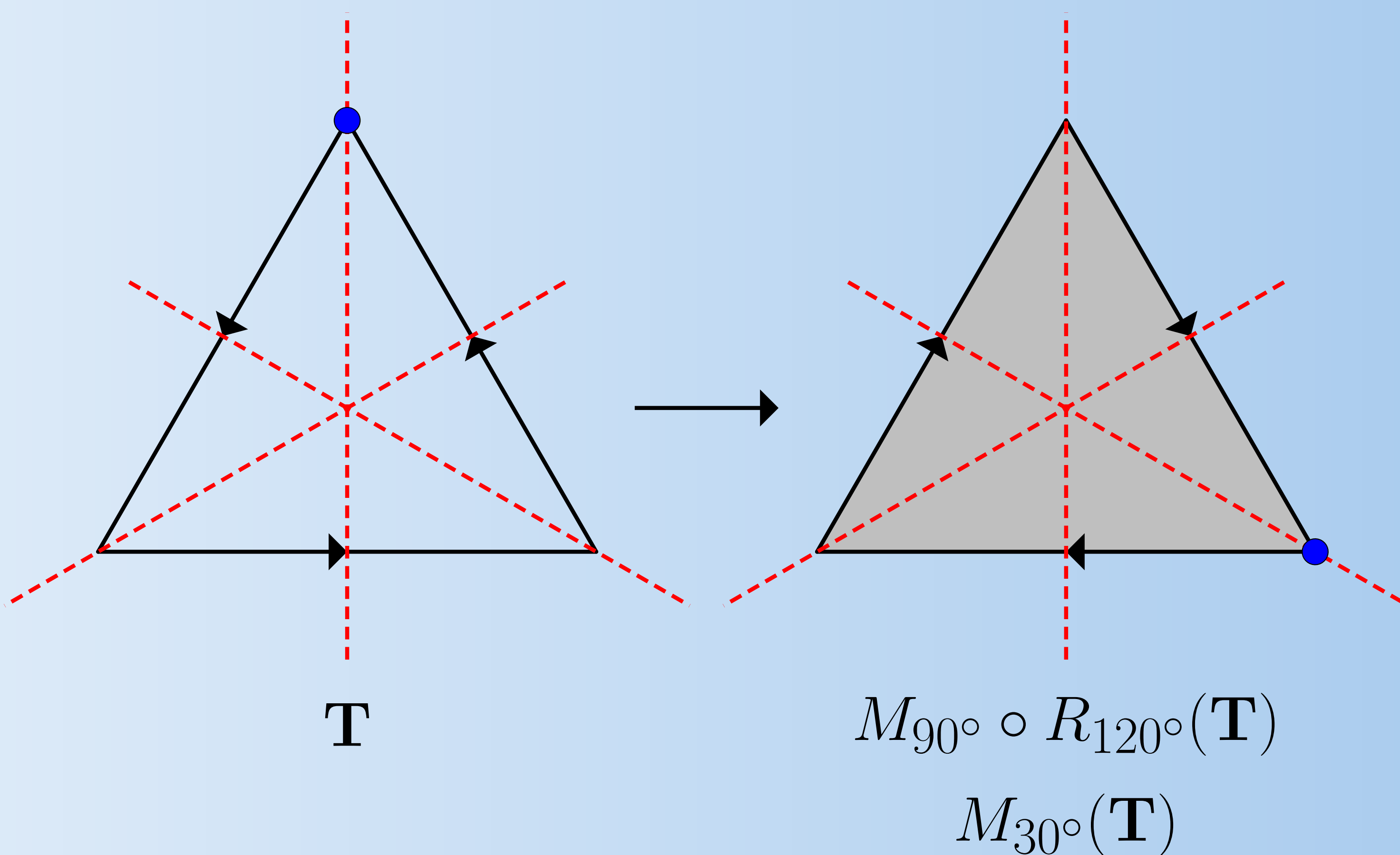


$\mathbf{T}$

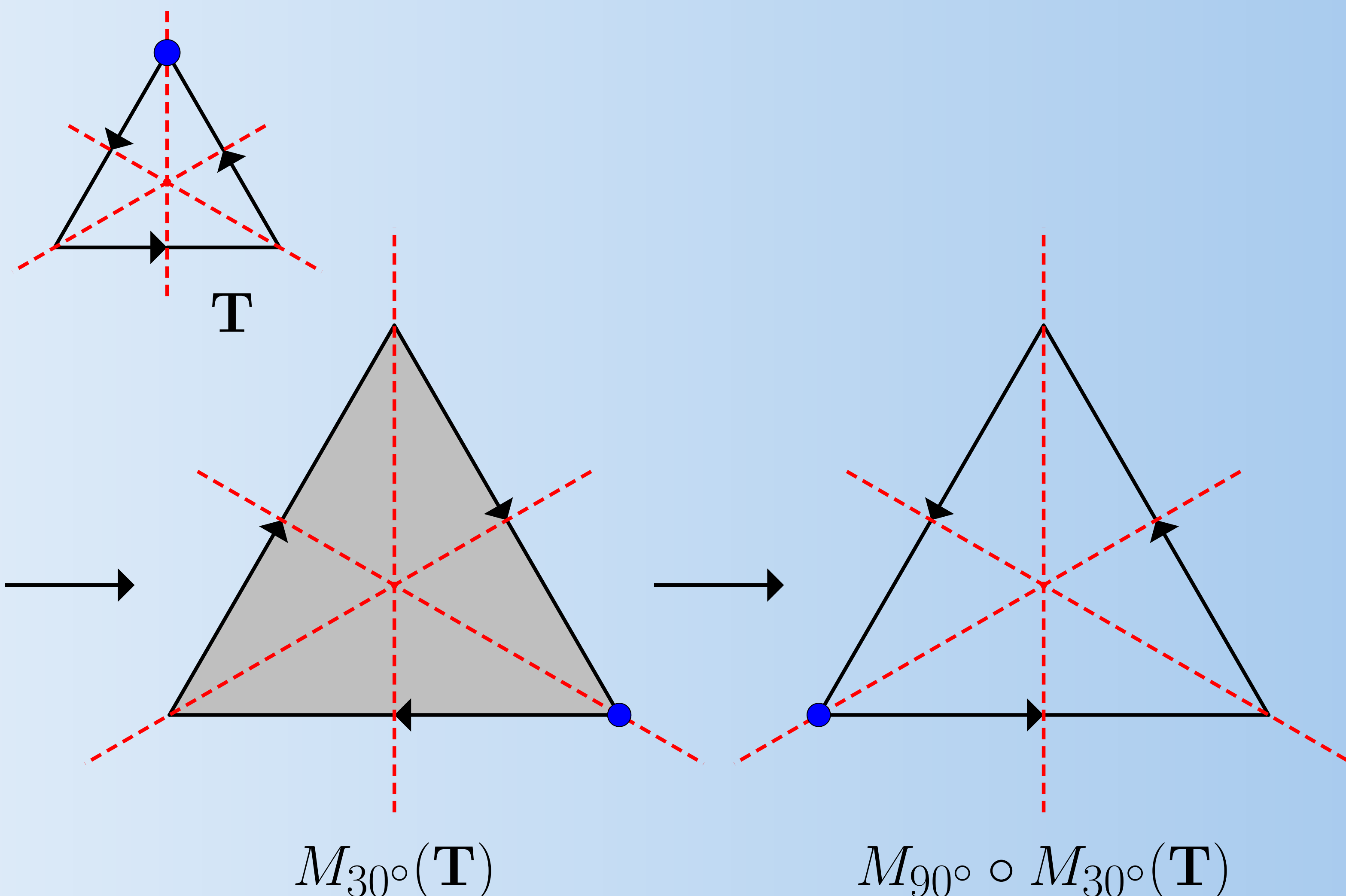
$M_{90^\circ}(R_{120^\circ}(\mathbf{T}))$

$M_{90^\circ} \circ R_{120^\circ}(\mathbf{T})$

But this has the same effect as M_{30°. Thus, we say that $M_{90^\circ} \circ R_{120^\circ} = M_{30^\circ}$.



Suppose we first reflect the triangle using M_{30°, and then reflect it with M_{90°. We see that $M_{90^\circ} \circ M_{30^\circ} = R_{120^\circ}$.



Tabulating Results

$\circ$	I	R_{120°	R_{240°	M_{30°	M_{90°	M_{120°
I	I	R_{120°	R_{240°	M_{30°	M_{90°	M_{120°
R_{120°	R_{120°					
R_{240°	R_{240°					
M_{30°	M_{30°					
M_{90°	M_{90°	M_{30°		R_{120°		
M_{120°	M_{120°					

The Completed Diagram

$\circ$	I	R_{120°	R_{240°	M_{30°	M_{90°	M_{120°
I	I	R_{120°	R_{240°	M_{30°	M_{90°	M_{120°
R_{120°	R_{120°	R_{240°	I	M_{90°	M_{120°	M_{30°
R_{240°	R_{240°	I	R_{120°	M_{120°	M_{30°	M_{90°
M_{30°	M_{30°	M_{120°	M_{90°	I	R_{240°	R_{120°
M_{90°	M_{90°	M_{30°	M_{120°	R_{120°	I	R_{240°
M_{120°	M_{120°	M_{90°	M_{30°	R_{240°	R_{120°	I

1. The same six transformations appear in each row and column.

If we denote the set of the transformations by

$$\mathcal{T} = \{I, R_{120^\circ}, R_{240^\circ}, M_{30^\circ}, M_{90^\circ}, M_{120^\circ}\},$$

we say that $\mathcal{T}$ is *closed under composition.*

Note that this does not fully explain why each of the six transformations appears *exactly once* in each row and column.

2. The identity transformation I appears in each row and column.

This is more significant, as it implies that every transformation has an inverse. For example, since

$$R_{90^\circ} \circ R_{270^\circ} = I,$$

R_{90° and then R_{270° are inverses of each other. We write $R_{90^\circ}^{-1} = R_{270^\circ}$, or equivalently, $R_{270^\circ}^{-1} = R_{90^\circ}$.

This helps explain why explain why each of the six transformations appears exactly once in each row and column.

Observation 2 (continued)

2. Suppose a transformation appeared twice in row M_{30°. Then there would be two transformations T_1 and T_2 such that $M_{30^\circ} \circ T_1 = M_{30^\circ} \circ T_2$.

Then, since function composition is associative,

$$\begin{aligned} M_{30^\circ} \circ (M_{30^\circ} \circ T_1) &= M_{30^\circ} \circ (M_{30^\circ} \circ T_2), \\ (M_{30^\circ} \circ M_{30^\circ}) \circ T_1 &= (M_{30^\circ} \circ M_{30^\circ}) \circ T_1, \\ I \circ T_1 &= I \circ T_2, \\ T_1 &= T_2. \end{aligned}$$

Thus, no row or column can contain duplicate transformations. Since each element in a row or column must contain an element of $\mathcal{T}$ without duplication in that row or column, each row or column must contain all six elements of $\mathcal{T}$.

The remarks in the first two observations imply that $\mathcal{T}$ has the structure of a *group.*

It is not important to go into details here, but they may be found in a book on abstract algebra.

This particular group is called the *dihedral group* D_3.

In general, the symmetries of a regular n-gon form the dihedral group D_n.

3. The composition of two reflections is a rotation.

We observed this fact geometrically; we will look at another explanation later.

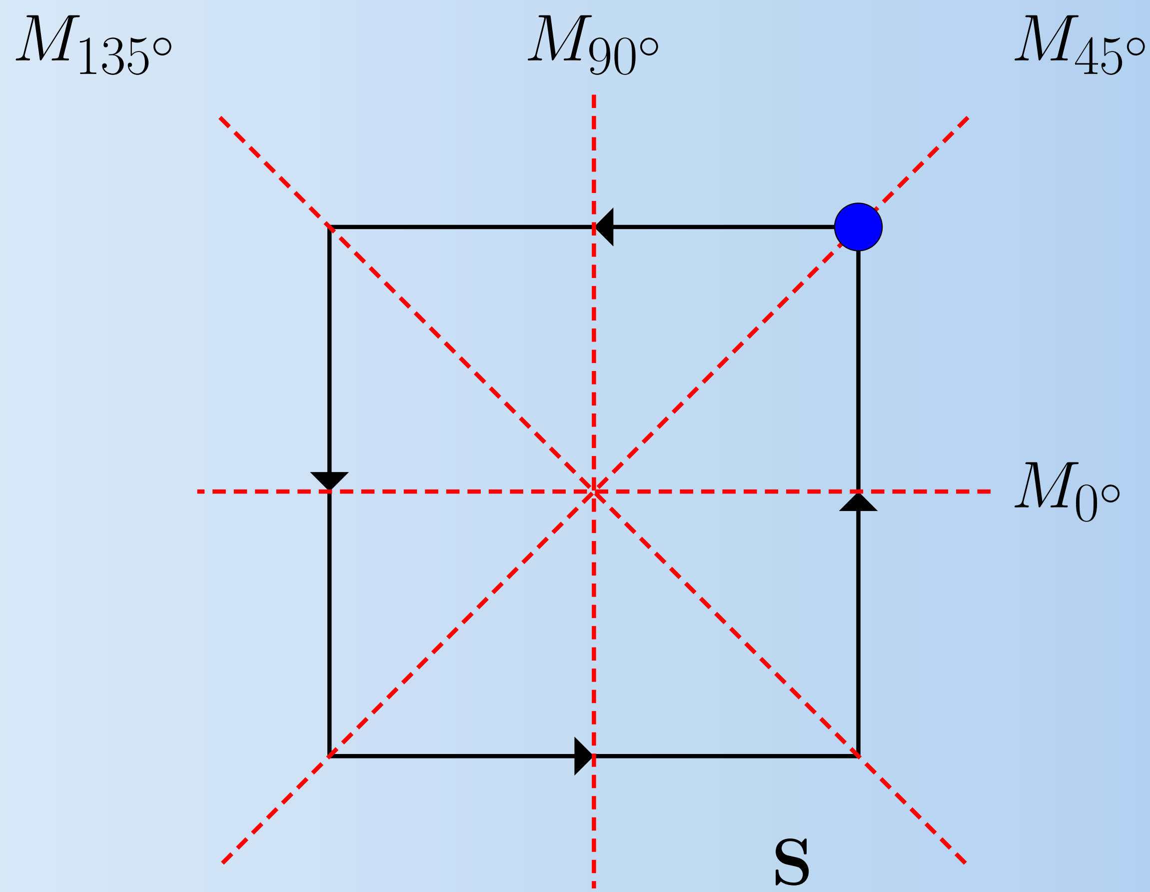
Note that this implies that the composition of a rotation and a reflection is a reflection. For example, if

$$M_{30^\circ} \circ M_{90^\circ} = R_{240^\circ},$$

then

$$\begin{aligned} M_{30^\circ} \circ R_{240^\circ} &= M_{30^\circ} \circ (M_{30^\circ} \circ M_{90^\circ}) \\ &= (M_{30^\circ} \circ M_{30^\circ}) \circ M_{90^\circ} \\ &= I \circ M_{90^\circ} \\ &= M_{90^\circ}. \end{aligned}$$

We continue studying the composition of reflections by exploring the symmetries of the square.



Compute the following compositions. How can we obtain the angle of rotation from the reflections?

$\circ$	M_{0°	M_{45°	M_{90°	M_{135°
M_{0°				
M_{45°				
M_{90°				
M_{135°				

How can we obtain the angle of rotation?

$\circ$	M_{0°	M_{45°	M_{90°	M_{135°
M_{0°	I	R_{270°	R_{180°	R_{90°
M_{45°	R_{90°	I	R_{270°	R_{180°
M_{90°	R_{180°	R_{90°	I	R_{270°
M_{135°	R_{270°	R_{180°	R_{90°	I

We see that for the composition of two reflections is given by

$$M_\alpha \circ M_\beta = R_{2(\alpha-\beta)}.$$

Note that this result is valid even when $\alpha - \beta < 0$. For example, our formula gives

$$M_{0^\circ} \circ M_{45^\circ} = R_{-90^\circ}.$$

But note that the rotation R_{-90° is the same as the rotation R_{270°.

Objectives:

- Review essential definitions.
- Understand how matrices describe geometric transformations.
- Create a matrix representation of a geometric transformation.
- Understand the significance of the determinant.
- Relate these ideas to the dihedral group.

A review of definitions:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

For all $\mathbf{v}$,

$$\mathbf{I}\mathbf{v} = \mathbf{v}.$$

This is because

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\begin{bmatrix} 1 & -2 \\ -1 & 5 \end{bmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 8 \\ -17 \end{pmatrix}$$

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & -1 \end{bmatrix} &= \begin{bmatrix} 1 \cdot 3 + 2 \cdot 4 & 1 \cdot 0 + 2 \cdot (-1) \\ -1 \cdot 3 + 5 \cdot 4 & -1 \cdot 0 + 5 \cdot (-1) \end{bmatrix} \\ &= \begin{bmatrix} 11 & -2 \\ 17 & -5 \end{bmatrix} \end{aligned}$$

Perform the following calculations:

1. $\begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

2. $\begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 3 & -4 \end{bmatrix}$

Solution to Practice Problems

Perform the following calculations:

1. $\begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 21 \\ -7 \end{pmatrix}$

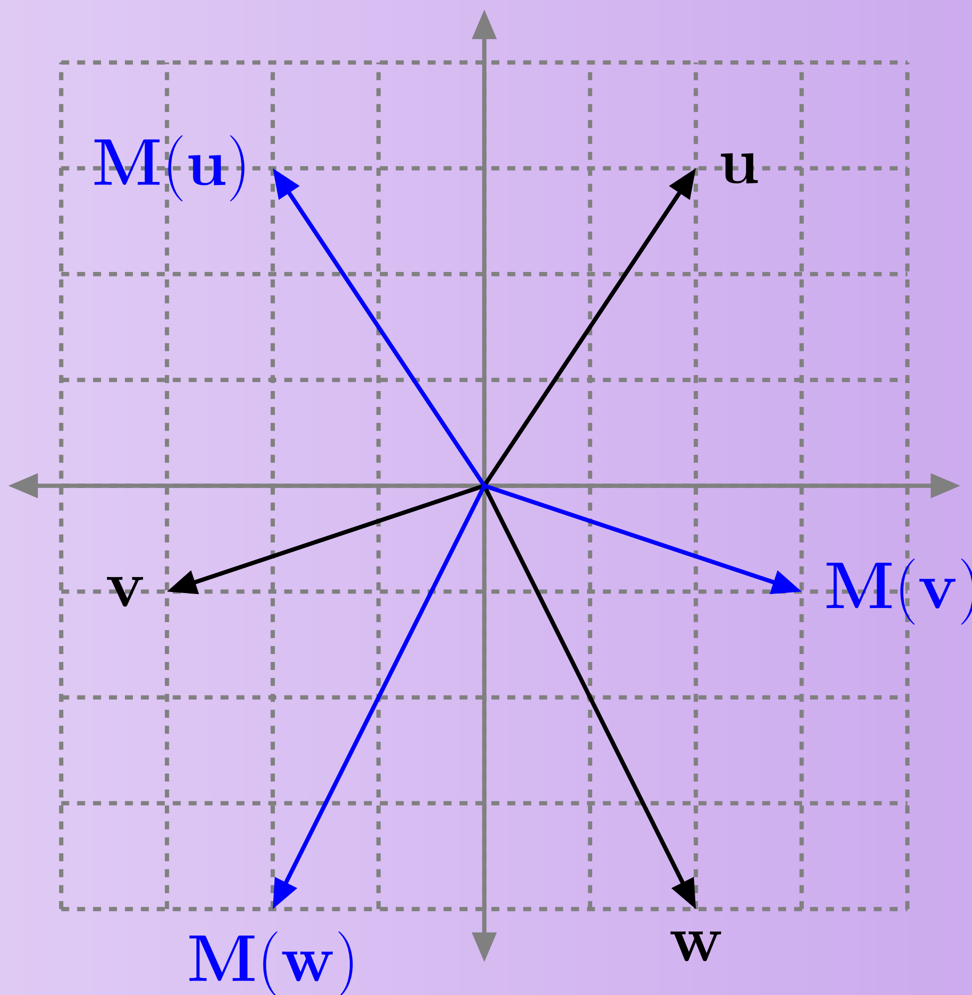
2. $\begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} -15 & 35 \\ 3 & -14 \end{bmatrix}$

Consider the effect of the matrix $\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ on various vectors.

$$\mathbf{M}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

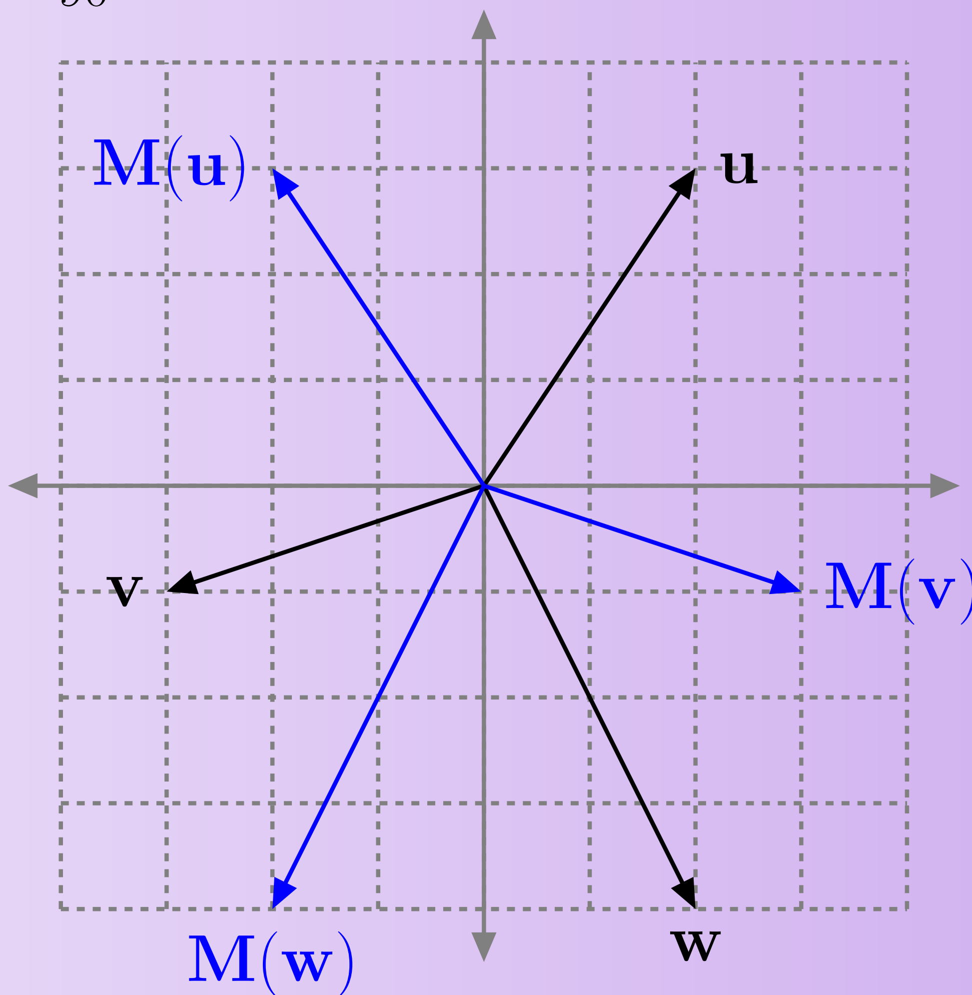
$$\mathbf{M}\begin{pmatrix} -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$$



Matrices Describe Reflections

We see that the matrix $\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ describes the reflection M_{90°.

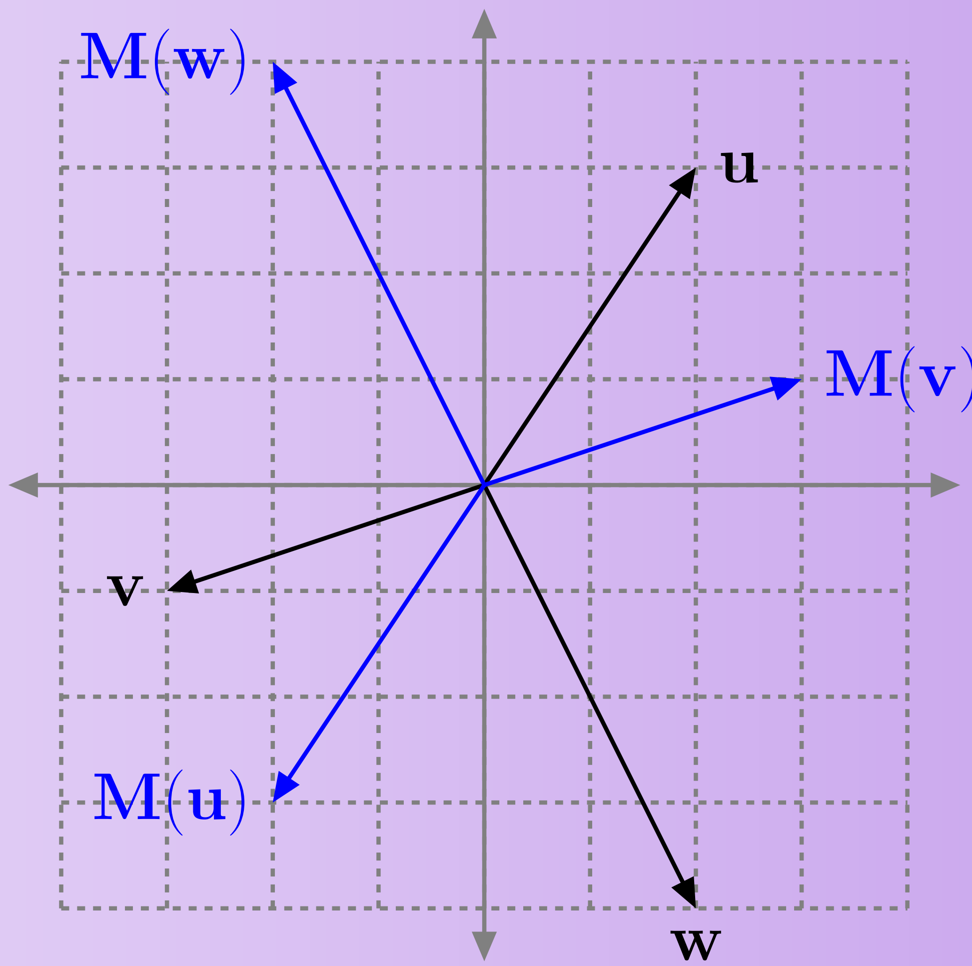


We see that the matrix $\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ describes the rotation R_{180° (a reflection through the origin).

$$\mathbf{M}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix} -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$



What geometric transformations do the following matrices describe?

1. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

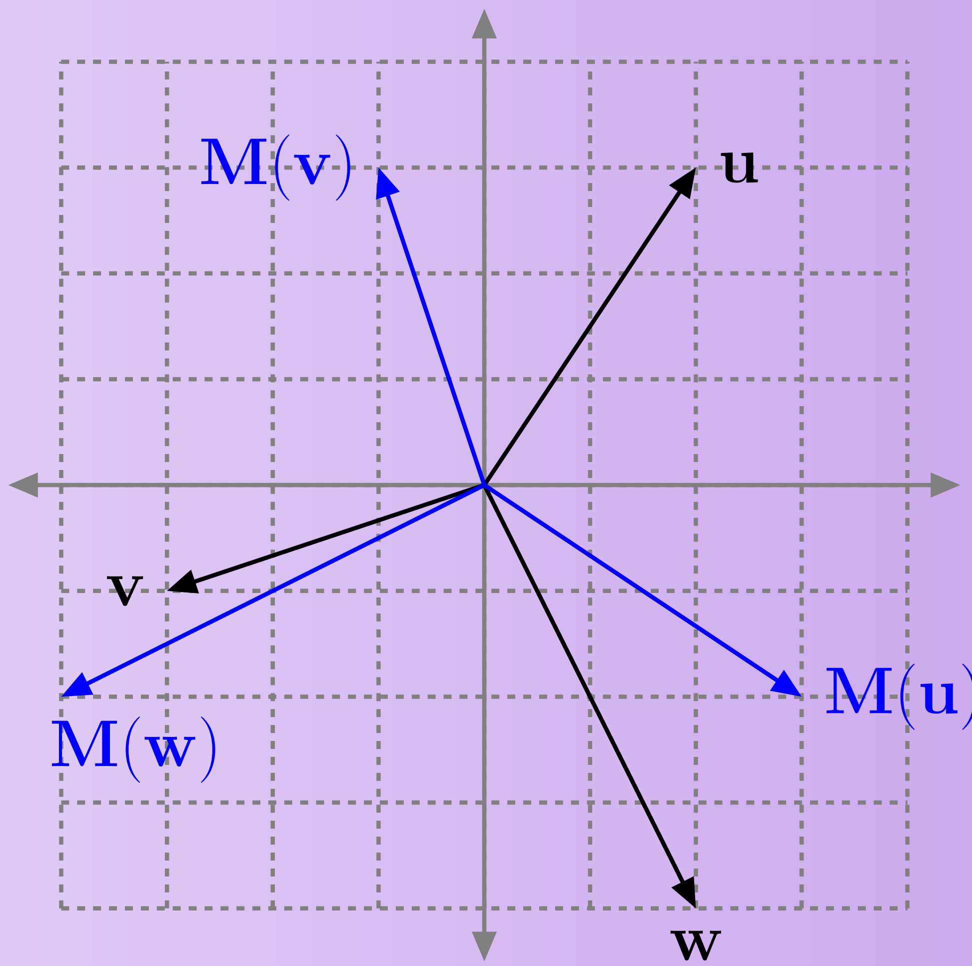
4. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

The matrix $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ describes the rotation R_{-90°, a clockwise rotation through 90°.

$$\mathbf{M}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix} -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

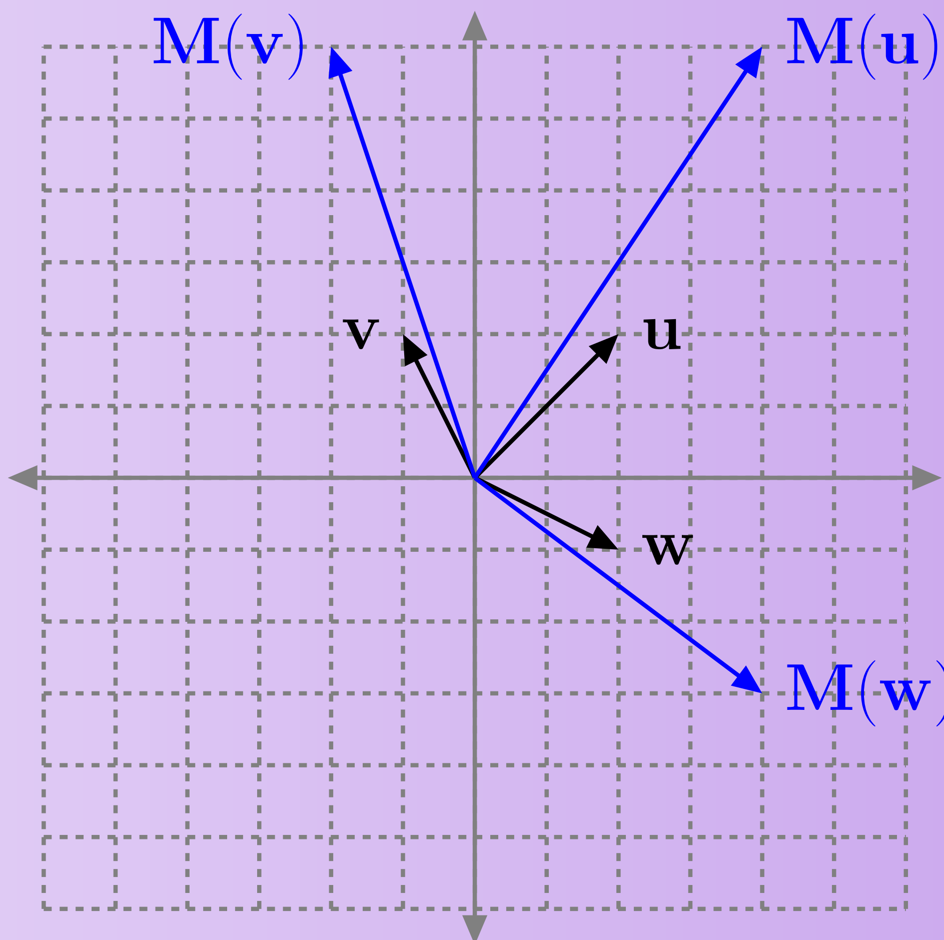


The matrix $\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ scales by a factor of 2 horizontally and a factor of 3 vertically.

$$\mathbf{M}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

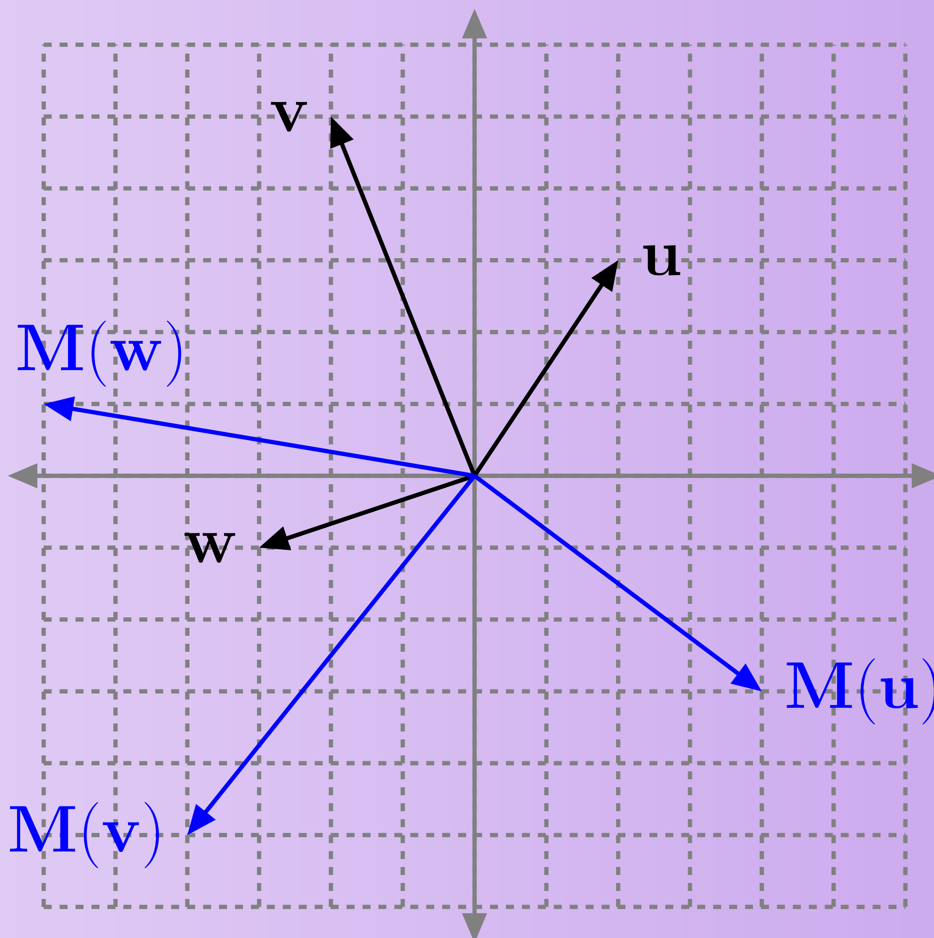


The matrix $\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ scales by a factor of 2 horizontally and reflects about the x-axis.

$$\mathbf{M}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix} -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$$



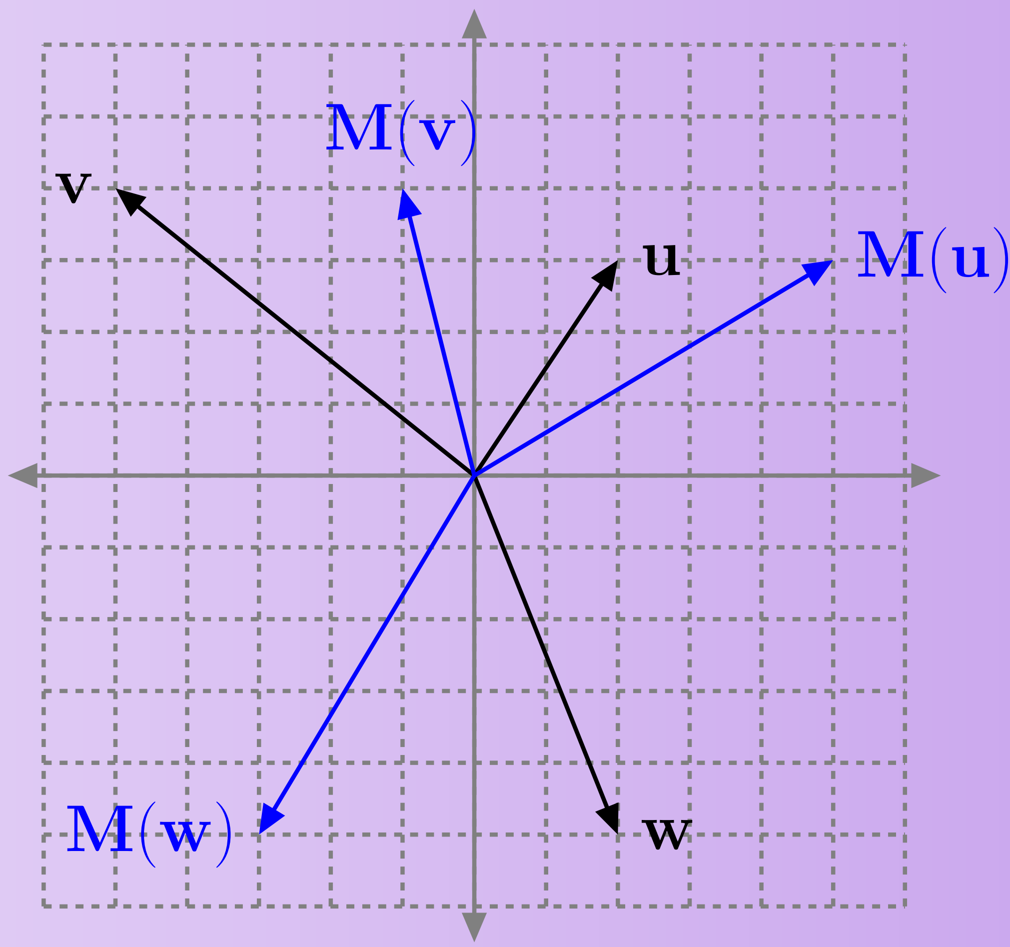
Example 4

The matrix $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *shears* horizontally. The y-coordinate stays the same, but affects the extent of the shear.

$$\mathbf{M}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix} -5 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$$



What does it mean to say that a matrix *describes* a geometric transformation?

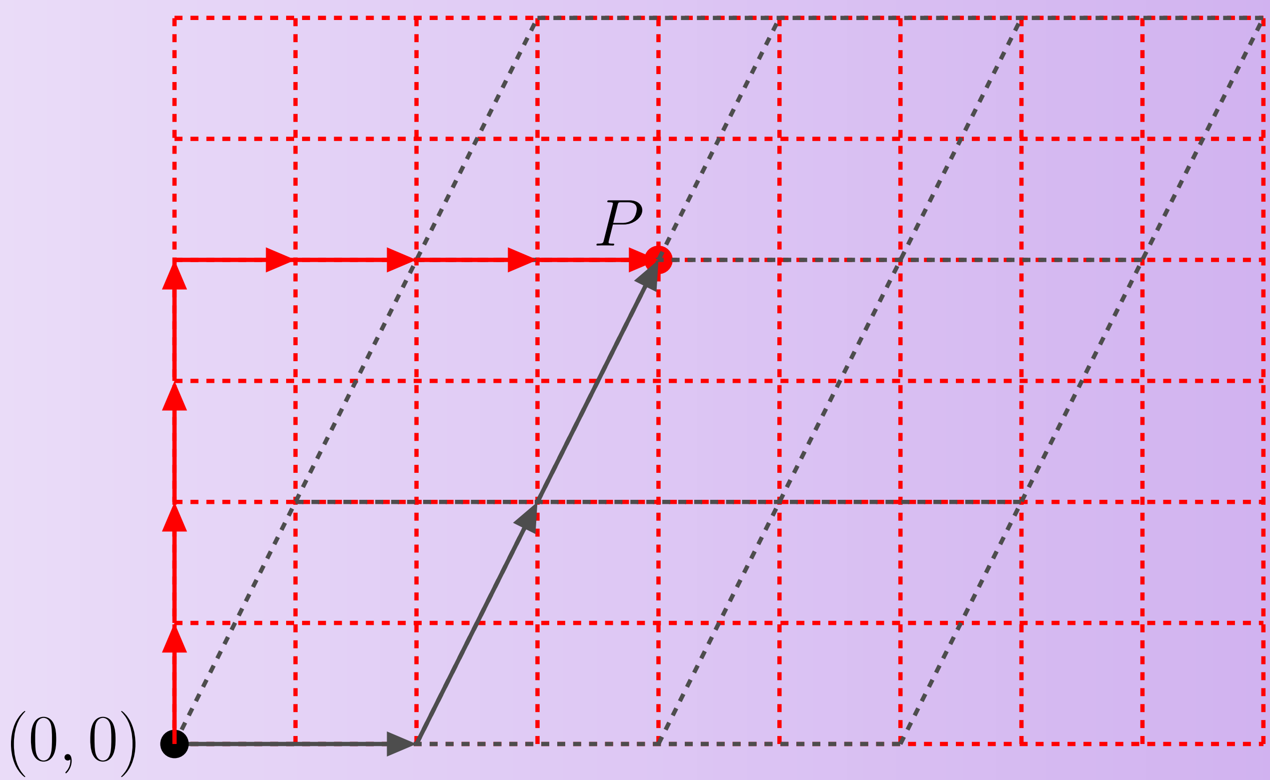
Symmetries of polygons may be defined purely geometrically.

To define a matrix, a *coordinate system* is needed.

Thus, a matrix can only be defined *relative* to a coordinate system. Thus, a matrix is simply one *representation* of a geometrical transformation.

Different coordinate systems produce different matrices.

In the rectangular (red) coordinate system, P has coordinates $(4,4)$. In the oblique (gray) coordinate system, P has coordinates $(1,2)$.



There is a convenient way of graphically representing a matrix. We may write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = x \begin{pmatrix} a \\ c \end{pmatrix} + y \begin{pmatrix} b \\ d \end{pmatrix}.$$

Also,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}.$$

Thus,

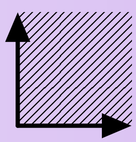
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Graphically Representing a Matrix

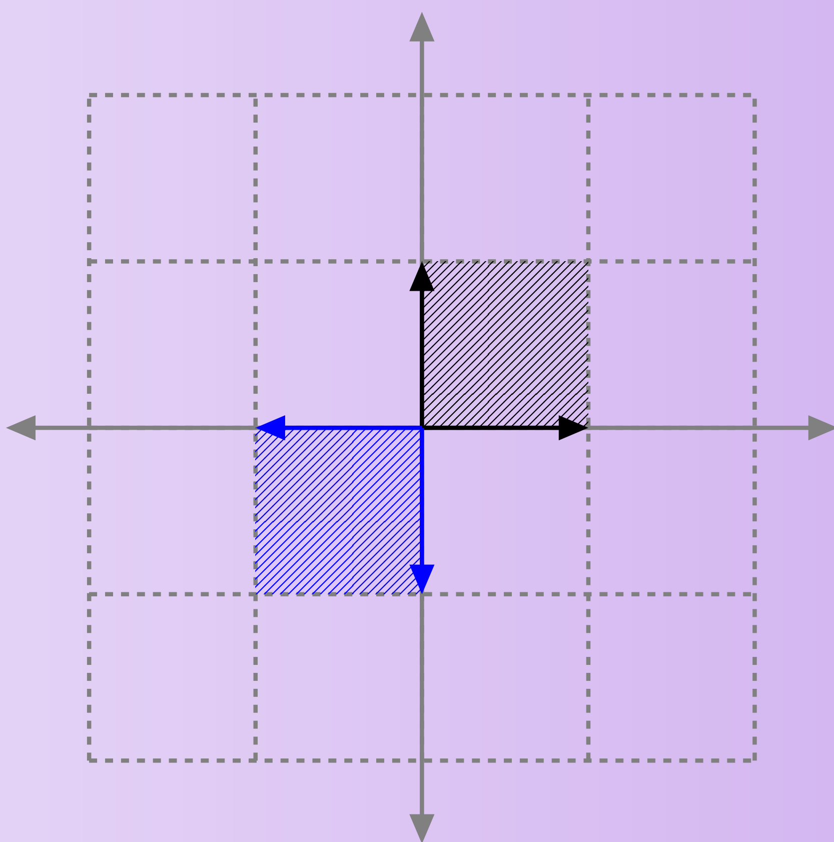
Thus, with $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = x\,\mathbf{M}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\,\mathbf{M}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

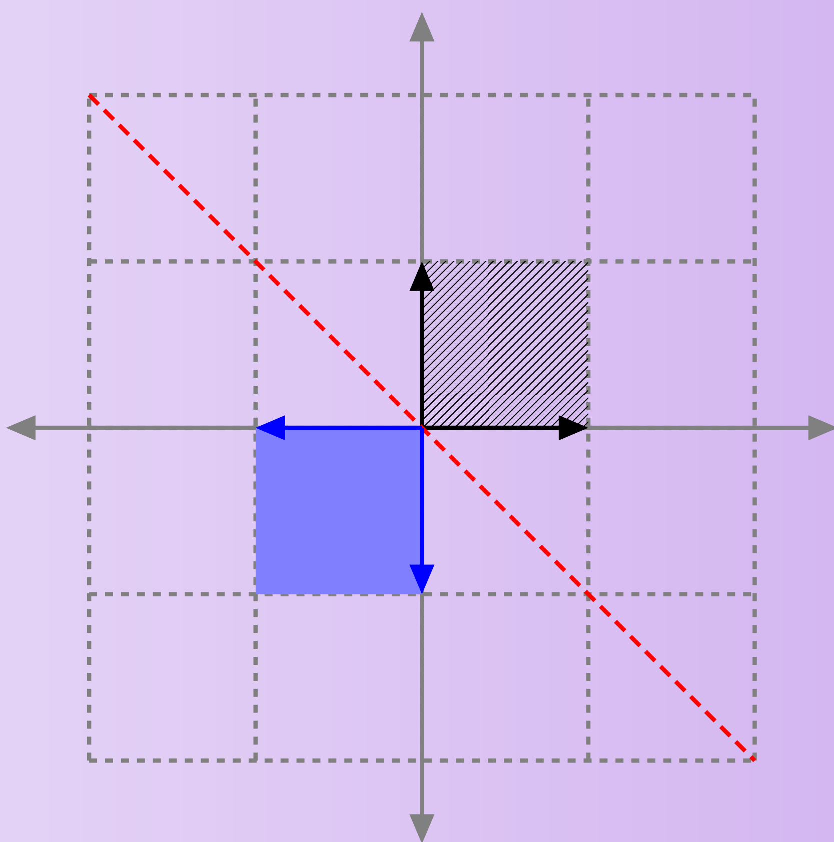
This means that just by knowing $\mathbf{M}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{M}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ – which are just the columns of the matrix – we can compute the value of $\mathbf{Mv}$ for all vectors, $\mathbf{v}$.

In other words, we can understand $\mathbf{M}$ by understanding how the unit square  is transformed.

The rotation R_{180° described by the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ may be represented by the following diagram:

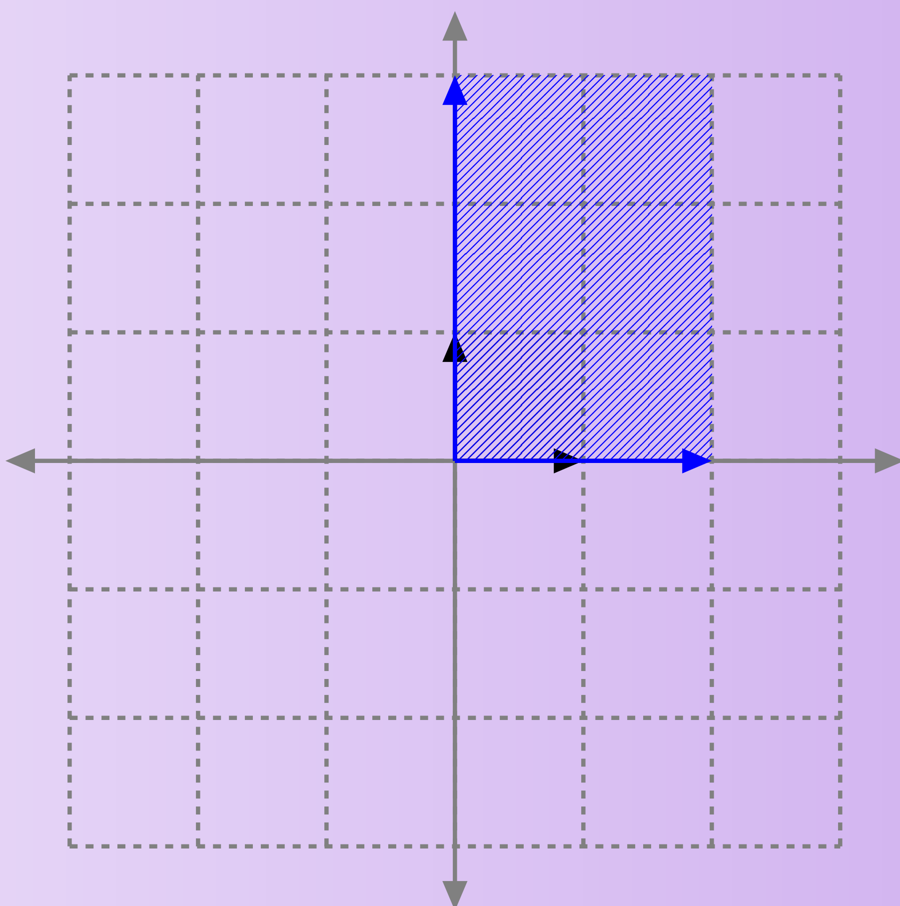


This can be distinguished from the reflection M_{135° described by the matrix $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ by filling in the square:



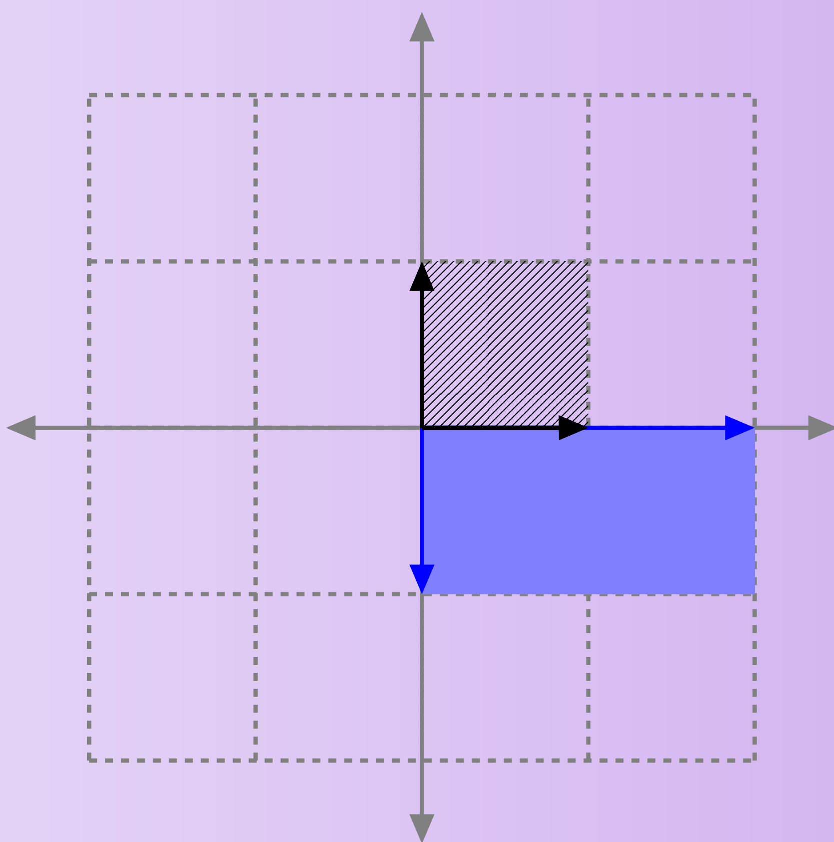
Example 2

The scaling described by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ may be represented by the following diagram:



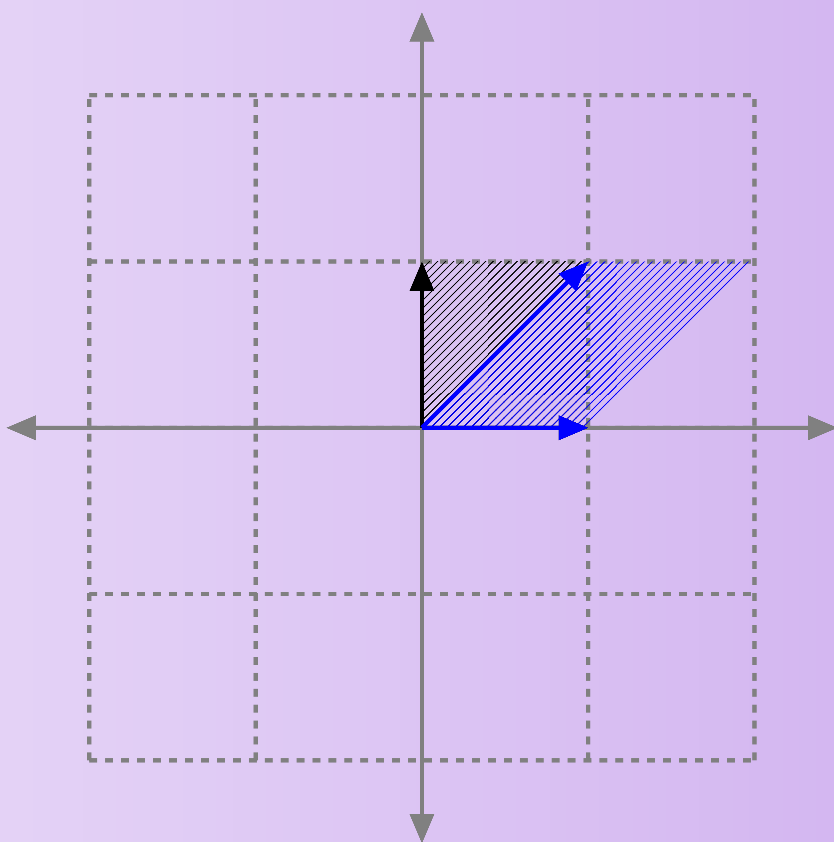
Example 3

The scaling and reflection described by $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ may be represented by the following diagram:



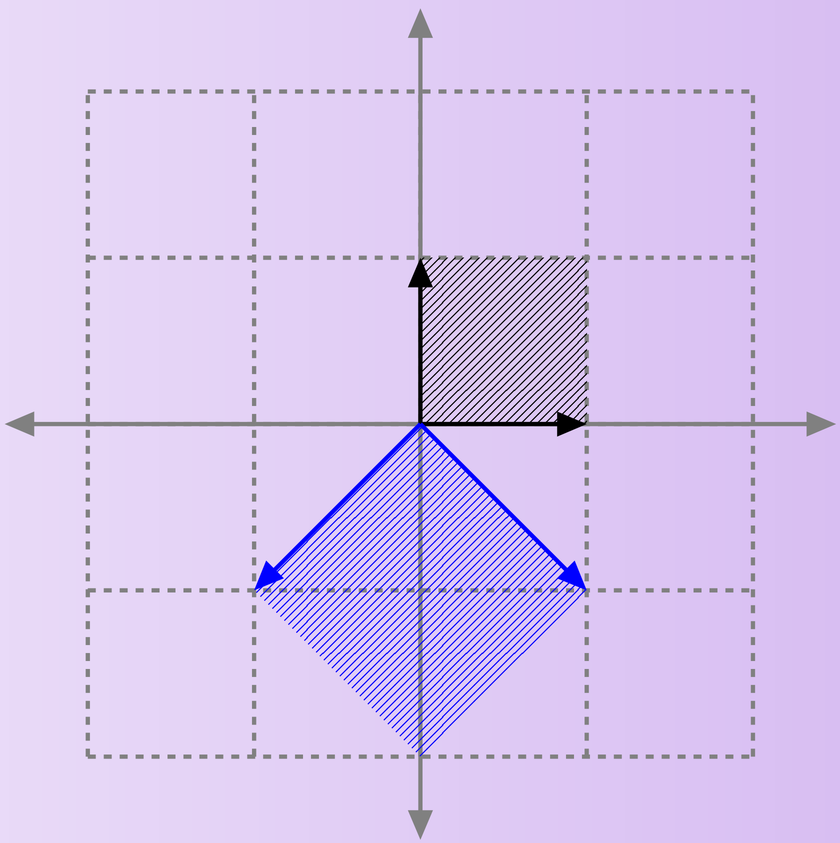
Example 4

The shear described by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ may be represented by the following diagram:



Finding a Matrix

Thus, if we know the effect of a transformation on the unit square, we may work backwards to find a matrix which describes the transformation.



$$\longrightarrow \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

Matrices of Square Symmetries

Using these ideas, find matrices which describe the symmetries of the square.

Matrices of Square Symmetries

The matrices below describe the symmetries of the square.

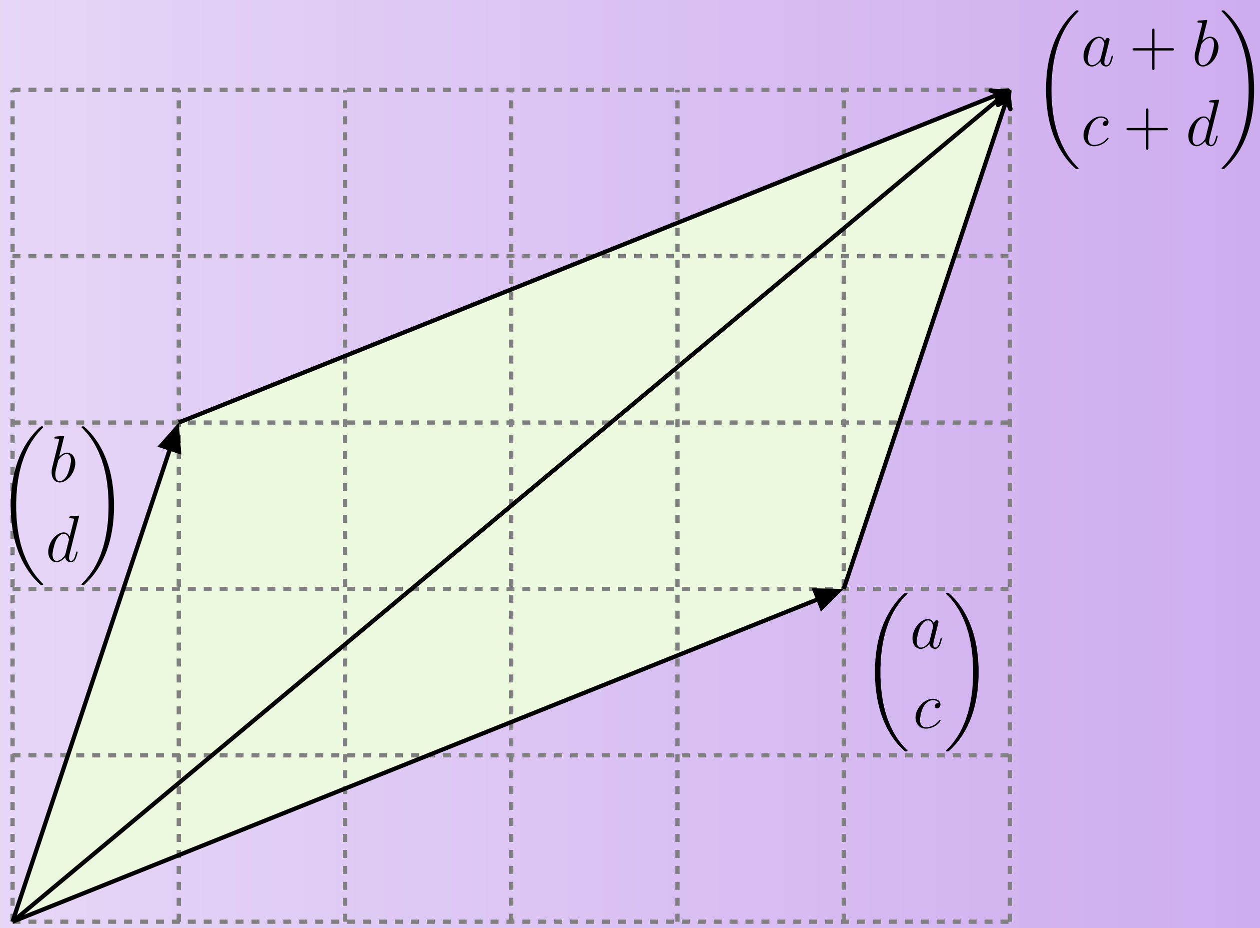
$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad M_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

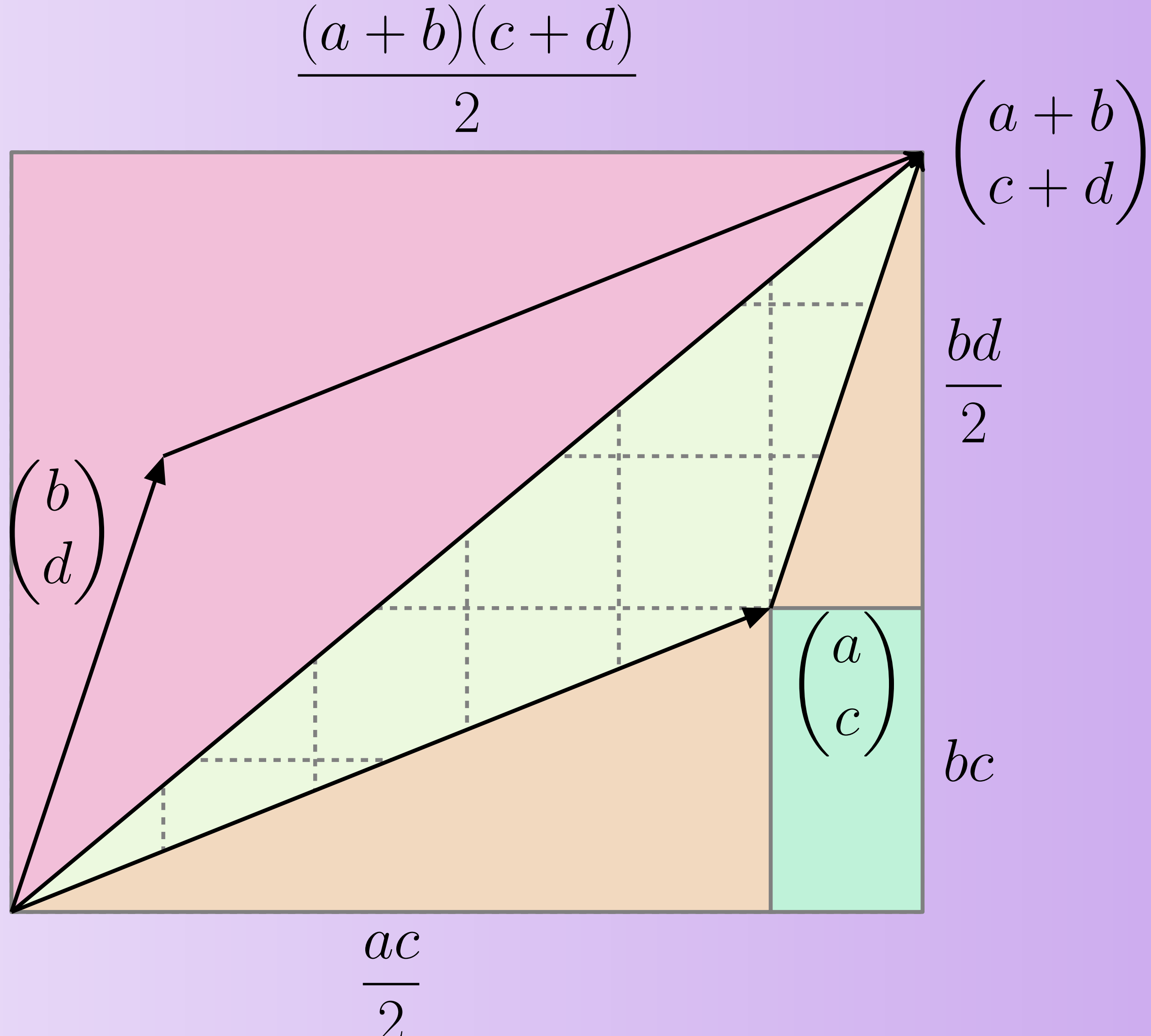
$$R_{90^\circ} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad M_{45^\circ} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$R_{180^\circ} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad M_{90^\circ} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

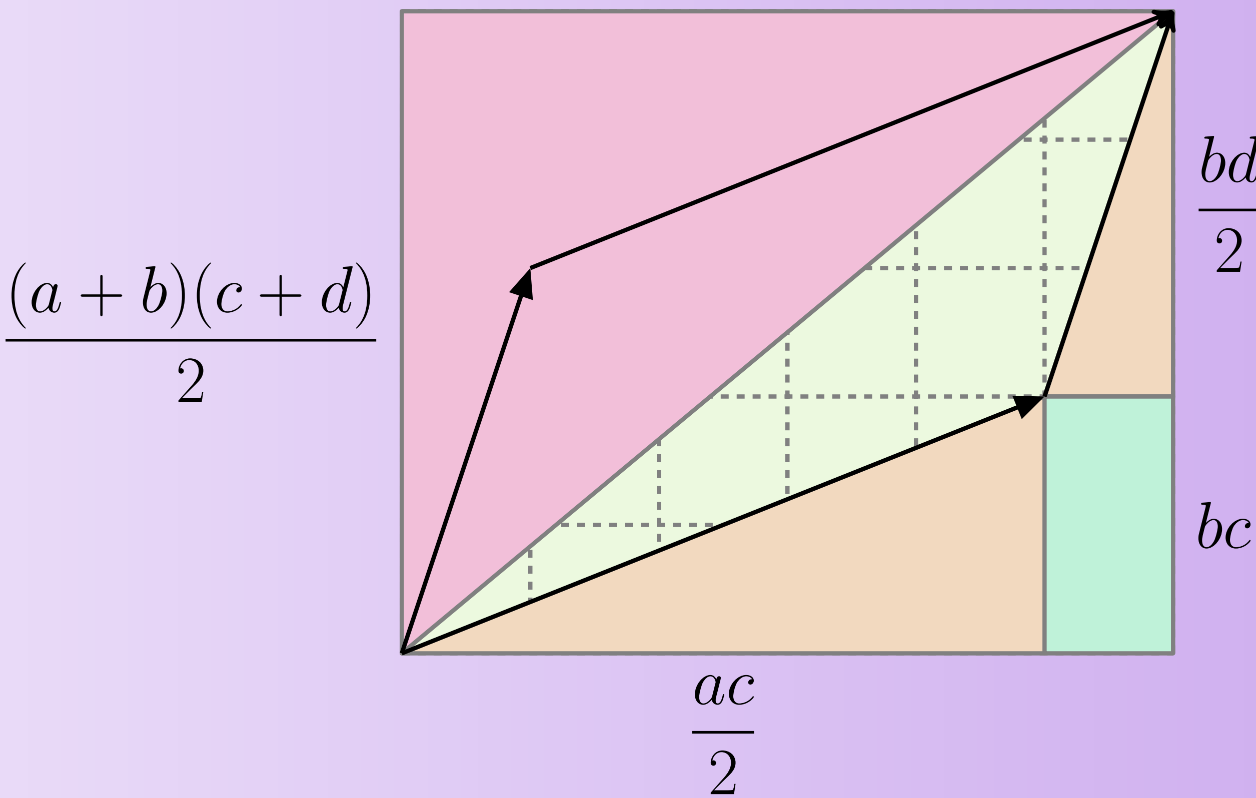
$$R_{270^\circ} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad M_{135^\circ} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

What is the area of the parallelogram?




$\frac{(a+b)(c+d)}{2}$
$\begin{pmatrix} a+b \\ c+d \end{pmatrix}$
$\begin{pmatrix} b \\ d \end{pmatrix}$
$\frac{bd}{2}$
$\begin{pmatrix} a \\ c \end{pmatrix}$
bc
$\frac{ac}{2}$

Calculating the Area

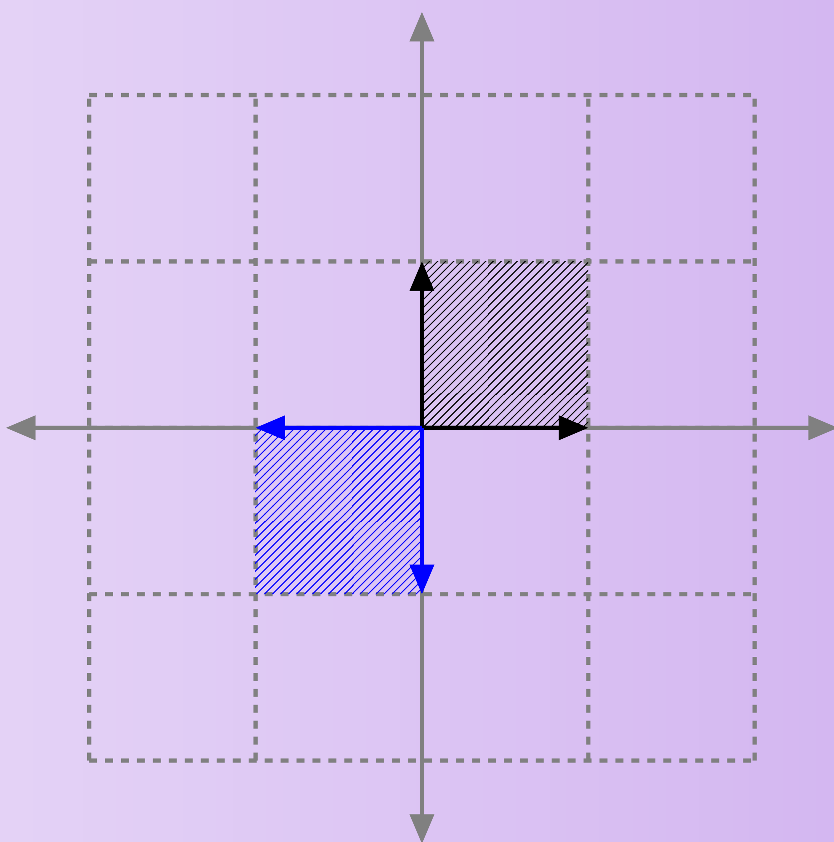


The area is:

$$2\left[\frac{(a+b)(c+d)}{2} - \frac{ac}{2} - \frac{bd}{2} - bc\right] = ad - bc.$$

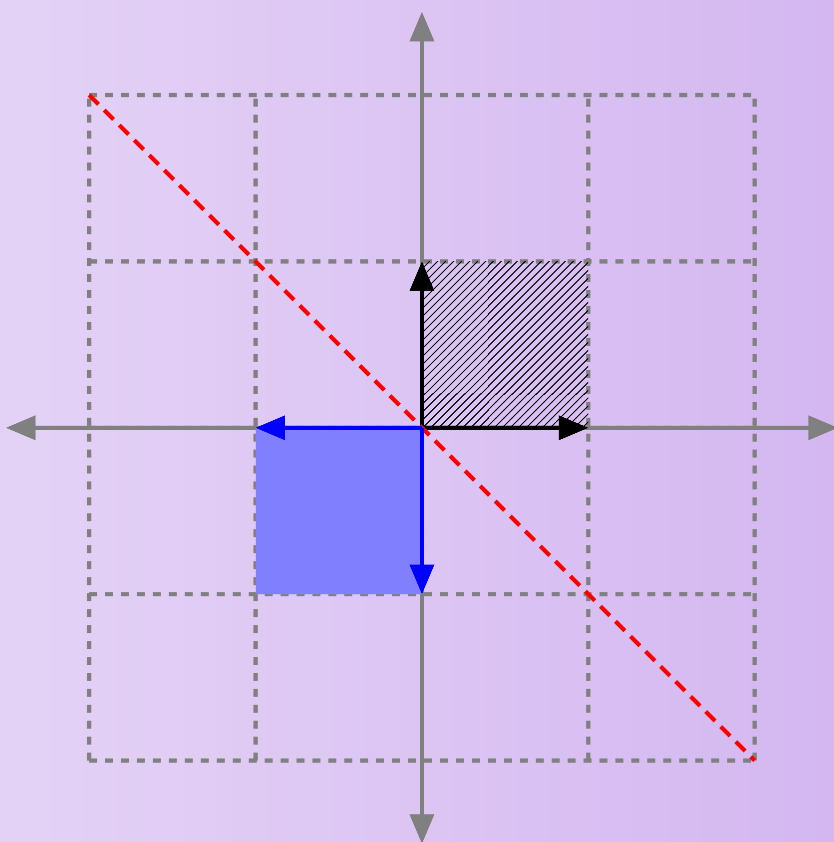
Graphical Representations, 1

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ has a determinant of $(-1) \cdot (-1) - 0 \cdot 0 = 1$, so that areas remain the same.



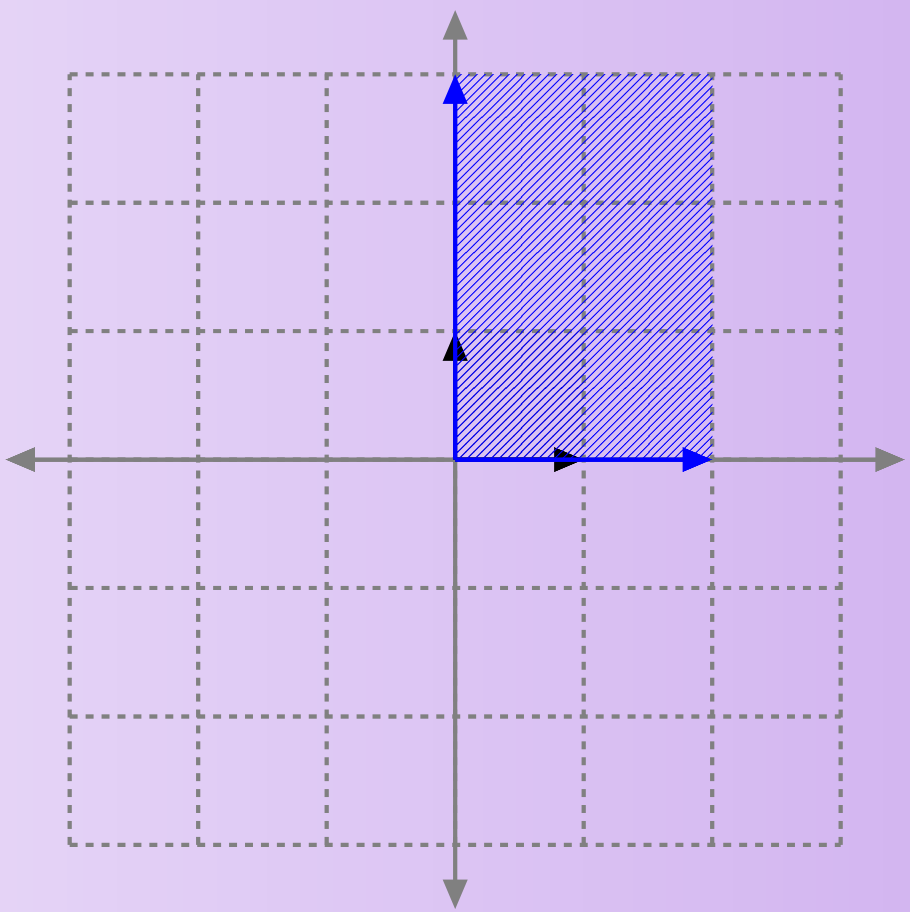
Graphical Representations, 2

The matrix $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ has determinant -1, so that area is preserved but a reflection is involved.



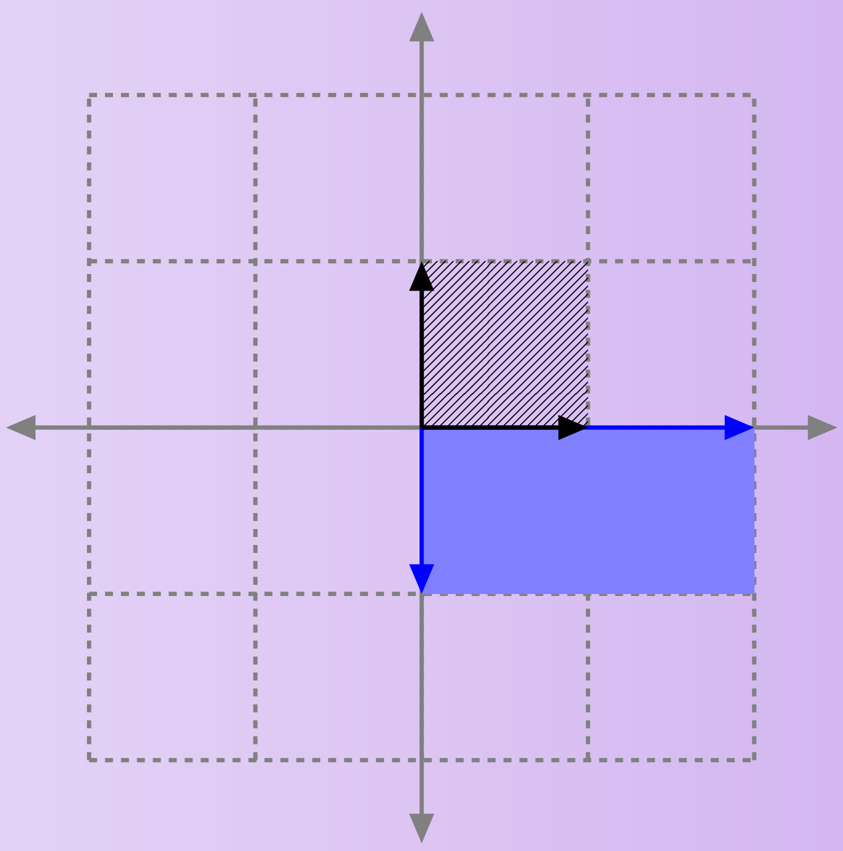
Example 2

The the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ has determinant 6, so that areas are increased by a factor of 6.



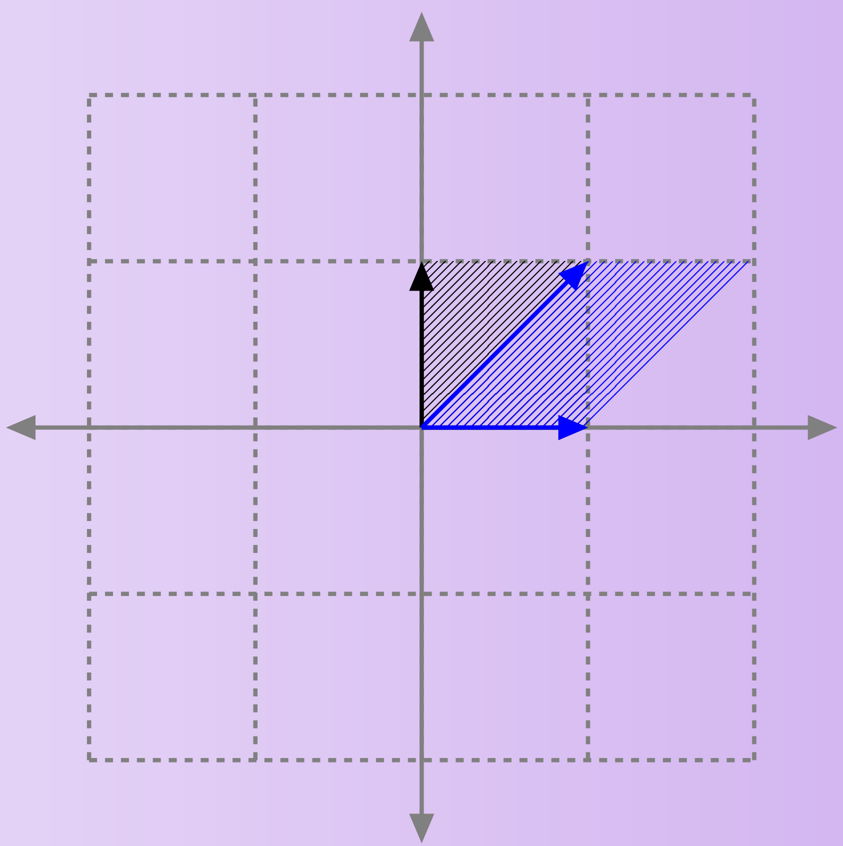
Example 3

The matrix $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ has determinant -2, so that areas are doubled and reflected.



Example 4

The shear described by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has determinant 1, so that areas remain the same.



Composition *vs.* Multiplication

If a transformation T_1 may be described by a matrix $\mathbf{M}_1$, and the transformation T_2 may be described by a matrix $\mathbf{M}_2$, the *composition* $T_2 \circ T_1$ may be described by the *product* $\mathbf{M}_2\mathbf{M}_1$ (where the product is the result of matrix *multiplication*).

Sometimes, we call the matrix $\mathbf{M}_1$ the transformation when we are working with the matrices *only.*

Remember: Describing a transformation with a matrix requires specifying a coordinate system. If we are representing all of our transformations on a rectangular coordinate system, we may think of the matrix as the transformation.

A Simple Proof

If $\mathbf{M}_2 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $\mathbf{M}_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ describe T_2 and T_1,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$$= \begin{pmatrix} A(ax + by) + B(cx + dy) \\ C(ax + by) + D(cx + dy) \end{pmatrix}$$

$$= \begin{pmatrix} (Aa + Bc)x + (Ab + Bd)y \\ (Ca + Dc)x + (Cb + Dd)y \end{pmatrix}$$

$$= \begin{bmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Verifying Compositions

Returning to the composition of reflections of a square, verify a few of the entries in the table you made using the matrix description of the transformations and matrix multiplication. For example, $M_{0^\circ} \circ M_{45^\circ} = R_{270^\circ}$, and

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The matrix transformation T_1 alters areas by a factor of $\det(\mathbf{M}_1)$, and the matrix transformation T_2 alters areas by a factor of $\det(\mathbf{M}_2)$. Thus, the composition $T_2 \circ T_1$ alters areas by a factor of $\det(\mathbf{M}_2)\det(\mathbf{M}_1)$, so that

$$\det(\mathbf{M}_2\mathbf{M}_1) = \det(\mathbf{M}_2) \cdot \det(\mathbf{M}_1).$$

Note that reflections have determinant -1, while rotations have determinant 1.

Recall that we observed that the product of two reflections is a rotation. The determinants multiply:

$$(-1) \cdot (-1) = 1.$$

Recall that the inverse of a matrix (when it has an inverse) is given by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Thus, if a transformation T is described by a matrix $\mathbf{M}$, and if T has an inverse, then the inverse of T is described by the matrix $\mathbf{M}^{-1}$.

With the inverse of a matrix defined, we may explore *groups* of matrices. One such group is that of the matrices which describe the symmetries of the square.

We have seen that matrices describe rotations and reflections. But it is easy to see that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and so *any* transformation described by a matrix leaves the origin of the coordinate system fixed.

Consider a simple transformation: a translation. A translation certainly shifts the origin, and so we have an example of a transformation which *cannot* be described by a matrix.

Question: What types of transformations can be described by matrices?

Adopting a Convention

Up to now, we have made a careful distinction between a transformation and a matrix which represents it.

As we will be working heavily with matrices and matrix notation for some time, we adopt the convention of referring to a transformation by its matrix without saying "the matrix $\mathbf{M}$ which described the transformation T." We simply refer to the matrix (or transformation) $\mathbf{M}$.

This is common usage when the context is clear.

Matrices Describe Linear Transformations

Some straightforward algebra shows that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(k \begin{pmatrix} x \\ y \end{pmatrix} \right) = k \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and

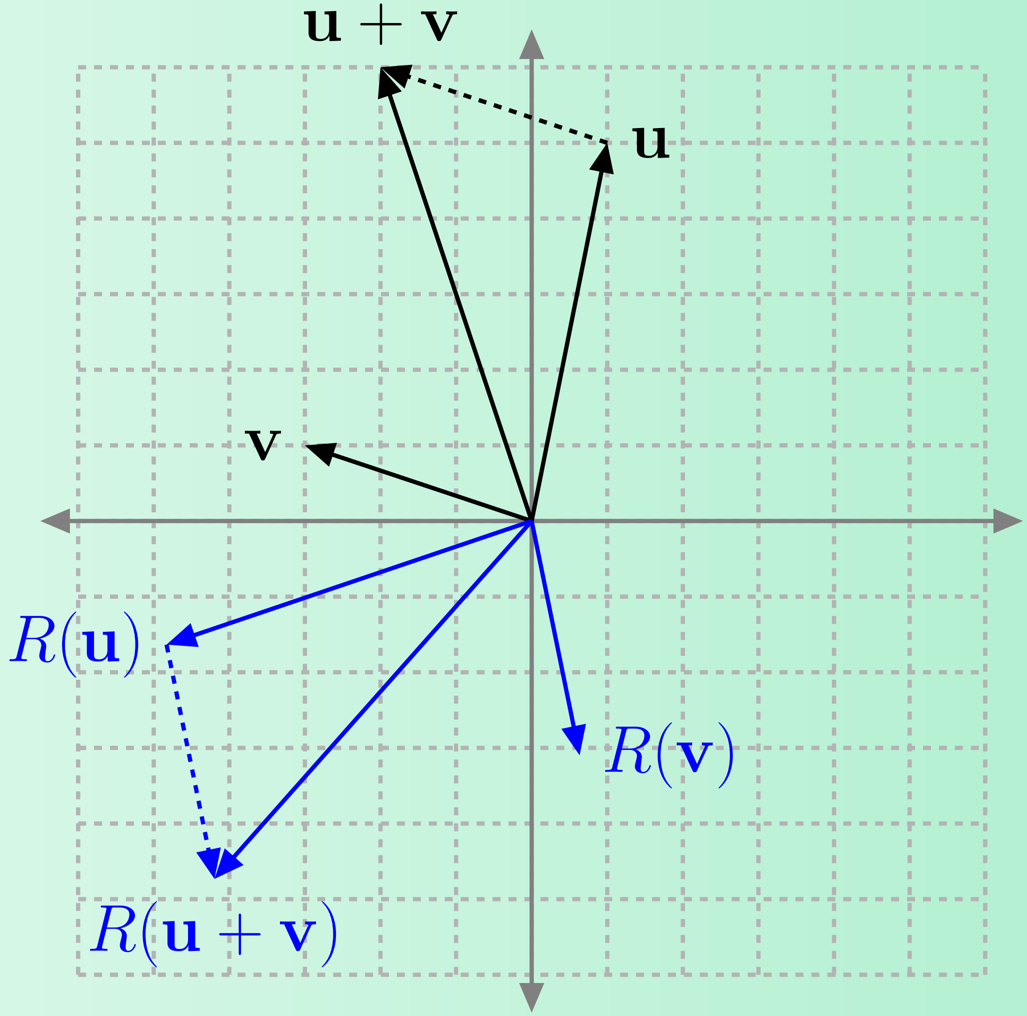
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right)$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Thus, matrices describe *linear transformations.* We know that linear transformations have a fixed point – the origin of the coordinate system. But linear transformations also preserve *linearity* – that is, they take lines into lines.

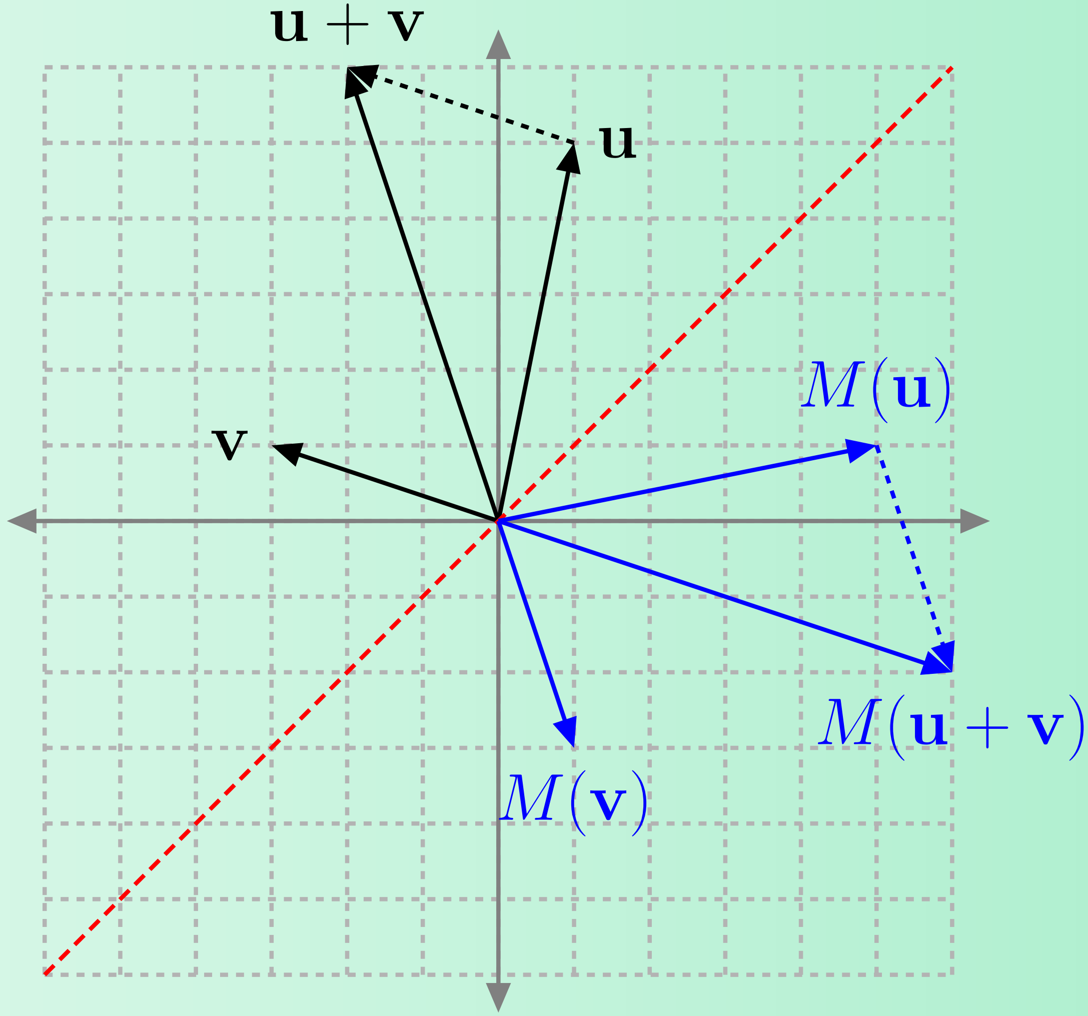
Rotations are Linear Transformations

A counterclockwise rotation by 120° is a linear transformation.



Reflections are Linear Transformations

A reflection about the line $y = x$ is a linear transformation.



In a rectangular coordinate system, consider what happens to the line $y = 3x - 2$ when transformed by $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$. We have

$$\begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} x \\ 3x - 2 \end{pmatrix} = \begin{pmatrix} 2 - x \\ 10x - 6 \end{pmatrix}.$$

Since $\widehat{x} = 2 - x$, then $x = 2 - \widehat{x}$, so that

$$\begin{aligned} \widehat{y} &= 10x - 6 \\ &= 10(2 - \widehat{x}) - 6 \\ &= 14 - 10\widehat{x}. \end{aligned}$$

Thus, the line $y = 3x - 2$ is transformed into the line $y = 14 - 10x$. This may be easily generalized.

Matrices Preserve Linearity

Here is an example to try.

What is the equation of the line obtained when $y = -x + 2$ is transformed by $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$?

What is the equation of the line obtained when $y = -x + 2$ is transformed by $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$?

We have

$$\begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{pmatrix} x \\ -x + 2 \end{pmatrix} = \begin{pmatrix} 4 - x \\ 8 - 5x \end{pmatrix}.$$

Since $\widehat{x} = 4 - x$, then $x = 4 - \widehat{x}$, so that

$$\begin{aligned} \widehat{y} &= 8 - 5x \\ &= 8 - 5(4 - \widehat{x}) \\ &= 5\widehat{x} - 12. \end{aligned}$$

Thus, the line $y = -x + 2$ is transformed into the line $y = 5x - 12$.

Translations Preserve Linearity

Recall that a translation is not linear (the origin is moved). But a translation *also* preserves lines. Consider the line $y = 3x - 2$ when transformed by the translation

$$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} x+2 \\ y-1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} = \mathbf{T}\begin{pmatrix} x \\ 3x-2 \end{pmatrix} = \begin{pmatrix} x+2 \\ 3x-3 \end{pmatrix}.$$

Since $\widehat{x} = x + 2$, then $x = \widehat{x} - 2$, and

$$\begin{aligned} \widehat{y} &= 3x - 3 \\ &= 3(\widehat{x} - 2) - 3 \\ &= 3\widehat{x} - 9. \end{aligned}$$

Note that the slope of the line did not change.

The most general transformation which preserves lines is the *affine transformation* given by

$$\mathbf{A}\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}e\\f\end{pmatrix}$$

$$= \mathbf{M}\begin{pmatrix}x\\y\end{pmatrix} + \mathbf{t}.$$

$\mathbf{M}$ is called the *linear part of* $\mathbf{A}$.

Suppose we wish to rotate 90° counterclockwise about the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. We proceed in three steps:

1. Translate $\begin{pmatrix} x \\ y \end{pmatrix}$ by $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ to $\begin{pmatrix} x-2 \\ y-1 \end{pmatrix}$. This moves $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to the origin.

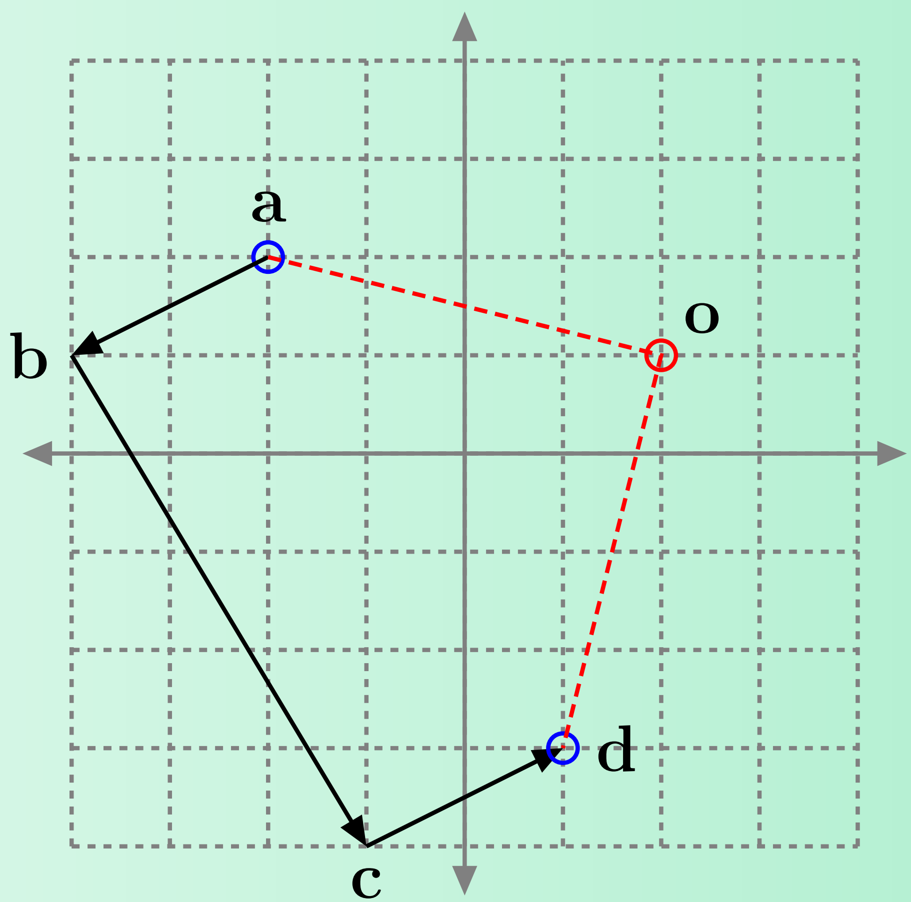
2. Now rotate:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x-2 \\ y-1 \end{pmatrix} = \begin{pmatrix} 1-y \\ x-2 \end{pmatrix}.$$

3. Now move back:

$$\begin{pmatrix} 1-y \\ x-2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-y \\ x-1 \end{pmatrix}.$$

$\mathbf{a}$ is translated to $\mathbf{b}$, then rotated 90° to $\mathbf{c}$, then translated back to $\mathbf{d}$. The effect is the same as rotating 90° about $\mathbf{o}$.



Rewriting the Affine Transformation

Recall that this transformation was represented by

$$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3-y \\ x-1 \end{pmatrix}.$$

We may rewrite this as:

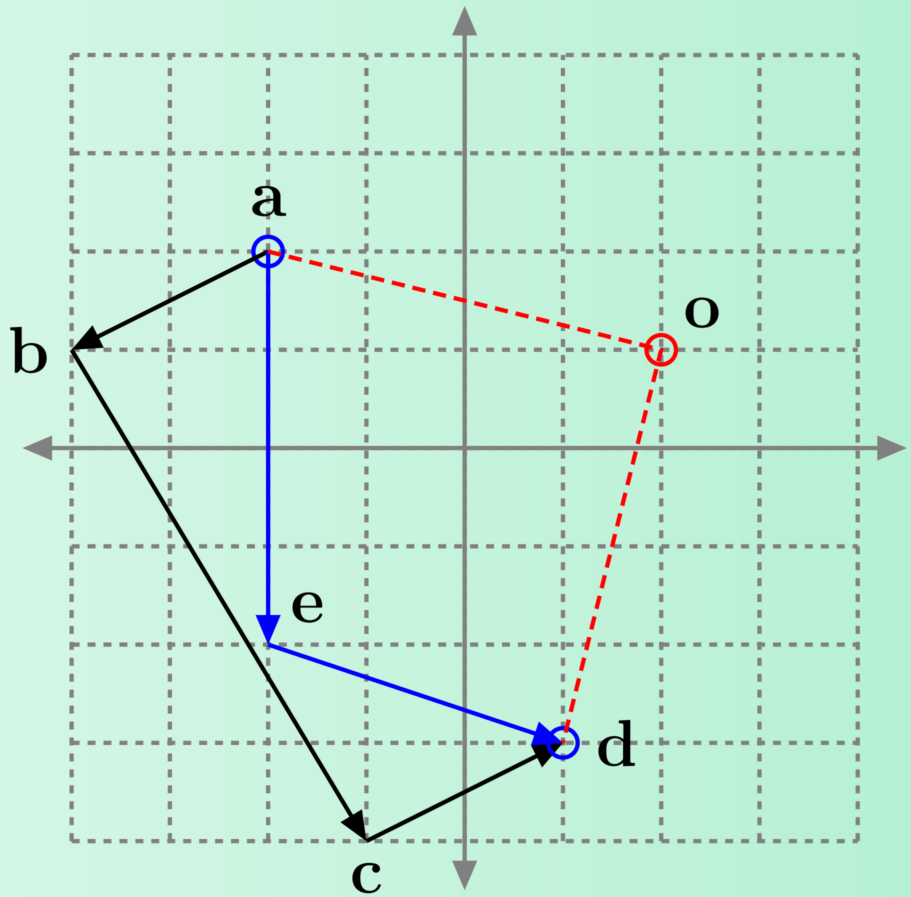
$$\begin{aligned} \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix}. \end{aligned}$$

Note that the linear part of this transformation is the rotation. This is because, in general,

$$\begin{aligned} \mathbf{Ax} &= \mathbf{M}(\mathbf{x}-\mathbf{t}) + \mathbf{t} \\ &= \mathbf{Mx} + (\mathbf{t} - \mathbf{Mt}). \end{aligned}$$

Graphical Representation, 2

As we rewrote the transformation, we may first rotate $\mathbf{a}$ 90° to $\mathbf{e}$, then translate by $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ to $\mathbf{d}$.



Find the affine transformation corresponding to a *clockwise* rotation of 90° about the point $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Use the relationship

$$\mathbf{A}\mathbf{x} = \mathbf{M}\mathbf{x} + (\mathbf{t} - \mathbf{M}\mathbf{t}),$$

where $\mathbf{M}$ is the linear part of $\mathbf{A}$.

Solution to Practice Problem

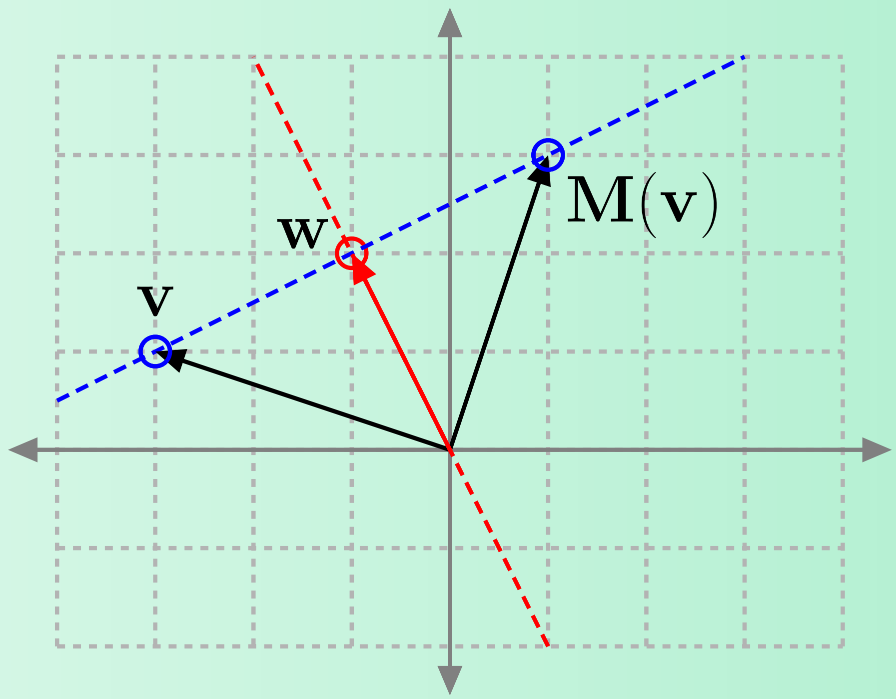
Find the affine transformation corresponding to a *clockwise* rotation of 90° about the point $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Solution:

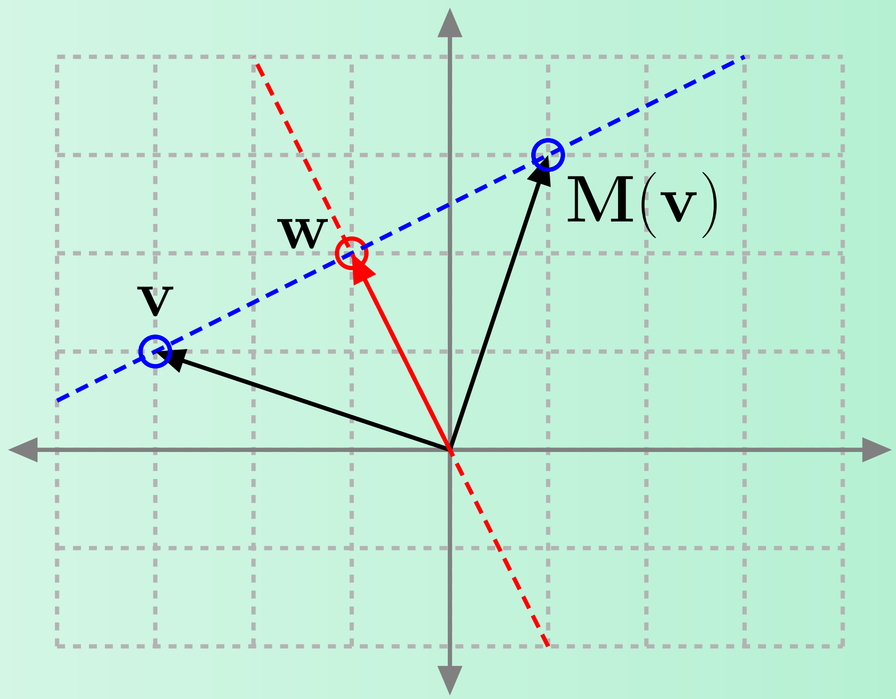
$$
\begin{aligned}
\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} &= \mathbf{Mx} + (\mathbf{t} - \mathbf{Mt}) \\
&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \end{pmatrix}.
\end{aligned}
$$

Problem: Reflect $\mathbf{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ about the line $y = -2x$.

Solution: The perpendicular line through $\mathbf{v}$ intersects the line at $\mathbf{w}$, the midpoint of the segment joining $\mathbf{v}$ and $\mathbf{M}(\mathbf{v})$.



The perpendicular line through $\mathbf{v}$, $y = \frac{1}{2}x + \frac{5}{2}$, intersects the line at $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. If $\frac{1}{2}\left(\begin{pmatrix} -3 \\ 1 \end{pmatrix} + \mathbf{M}(\mathbf{v})\right) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, then the reflection of $\mathbf{v}$ is $\mathbf{M}(\mathbf{v}) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.



Using an Arbitrary Point

Working through the same procedure with an arbitrary point $\begin{pmatrix} x \\ y \end{pmatrix}$, we find that

$$\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Again working through the same procedure, this time reflecting the point $\begin{pmatrix} x \\ y \end{pmatrix}$ about the line $y = mx$, we find that

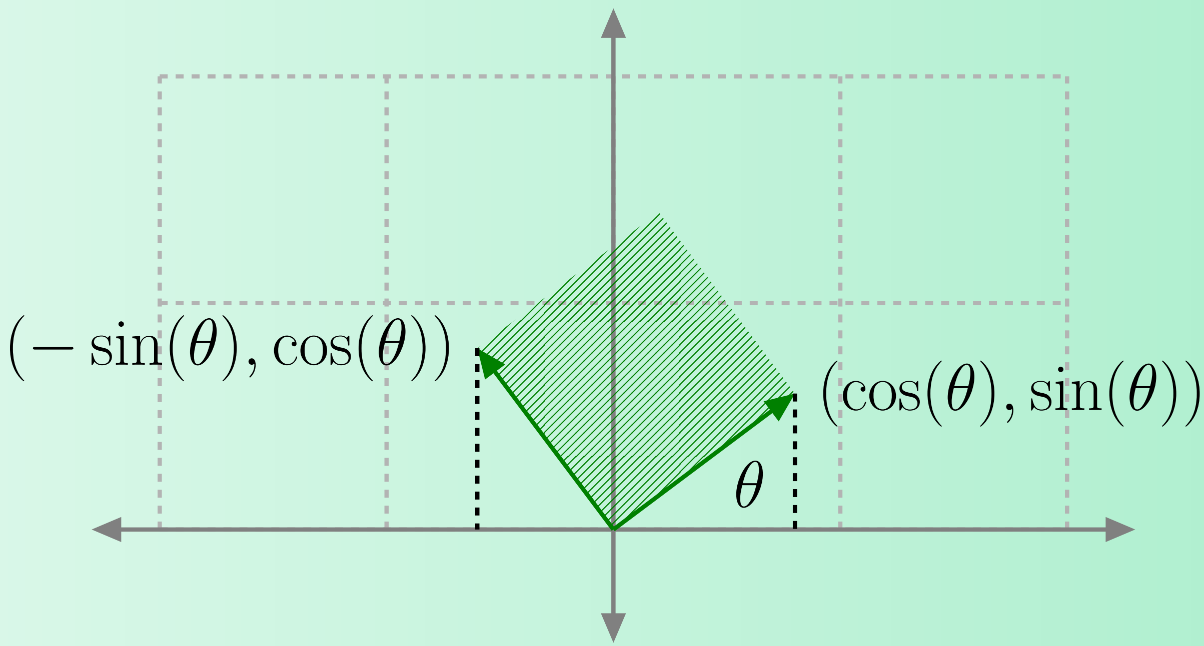
$$\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2} \\ \\ \dfrac{2m}{1+m^2} & \dfrac{m^2-1}{1+m^2} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

While most students will not have a knowledge of trigonometry, it is important to note now trigonometry may be used to obtain rotation and reflection matrices.

In the next three slides, we briefly demonstrate this.

The rotation matrix for a counterclockwise rotation through an angle θ is given by

$$\mathbf{R}_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$



The matrix for a reflection about a line making an angle θ with the positive x-axis is given by

$$\mathbf{M}_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \dfrac{1-\tan^2(\theta)}{1+\tan^2(\theta)} & \dfrac{2\tan(\theta)}{1+\tan^2(\theta)} \\ \\ \dfrac{2\tan(\theta)}{1+\tan^2(\theta)} & \dfrac{\tan^2(\theta)-1}{1+\tan^2(\theta)} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This is found by using $m = \tan(\theta)$ in the previous formula.

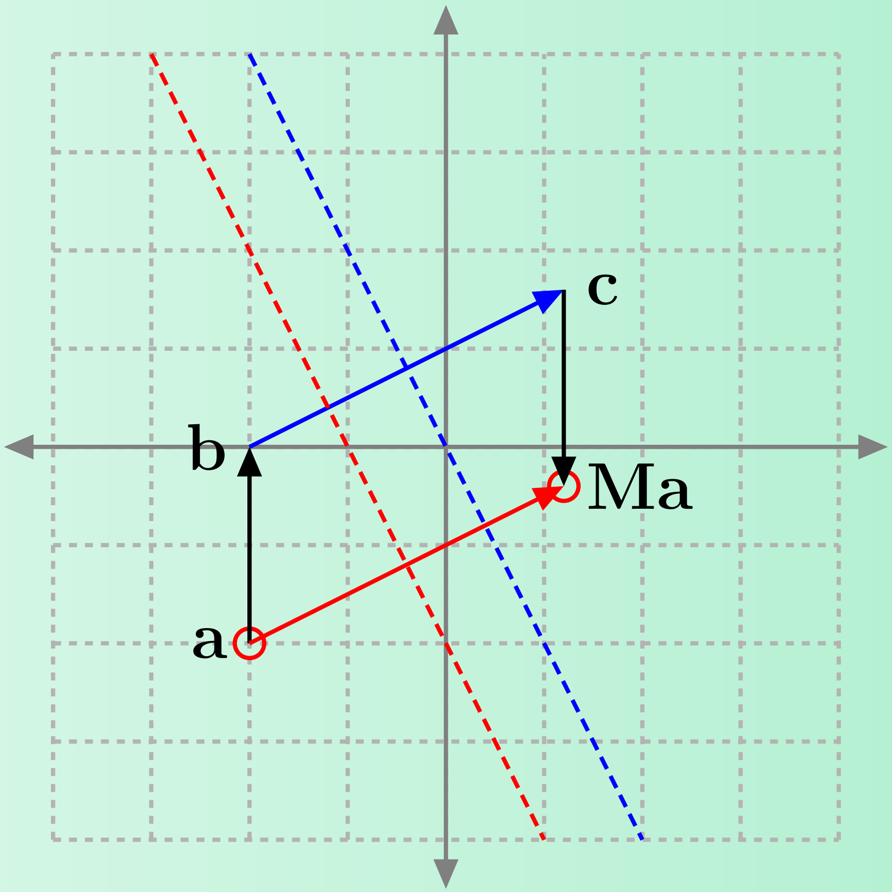
Reflecting about a line which makes an angle α with the positive x-axis, then about a line which makes an angle β with the positive x-axis, results in (using trigonometric identities as appropriate)

$$\mathbf{M}_\beta \mathbf{M}_\alpha = \begin{bmatrix} \cos(2\beta) & \sin(2\beta) \\ \sin(2\beta) & -\cos(2\beta) \end{bmatrix} \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2(\beta-\alpha)) & -\sin(2(\beta-\alpha)) \\ \sin(2(\beta-\alpha)) & \cos(2(\beta-\alpha)) \end{bmatrix}$$

$$= \mathbf{R}_{2(\beta-\alpha)}.$$

To reflect $\mathbf{a}$ about the line $y = -2x - 2$, first translate so the line passes through the origin, which takes $\mathbf{a}$ to $\mathbf{b}$. Then reflect as before to $\mathbf{c}$, then translate back to $\mathbf{Ma}$. (Note: Any translation will work here.)



We use the relationship $\mathbf{Ax} = \mathbf{M}(\mathbf{x} - \mathbf{t}) + \mathbf{t}$ to obtain

$$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{pmatrix} x \\ y - 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$= \begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{8}{5} \\ \frac{4}{5} \end{pmatrix}.$$

Practice Problem

Find the affine transformation corresponding to a reflection about the line $y = \frac{1}{2}x + 1$.

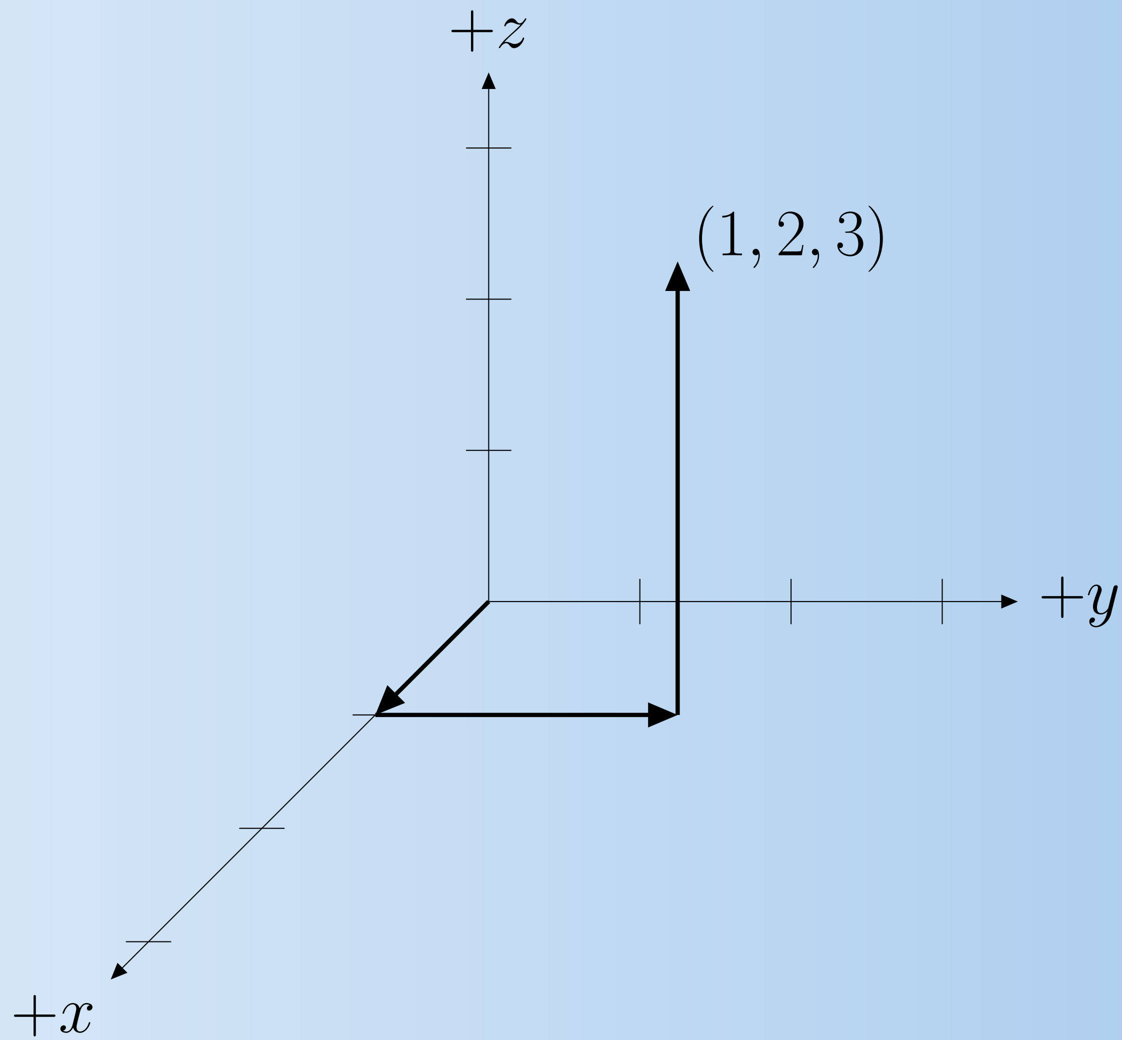
Solution to Practice Problem

We use the relationship $\mathbf{Ax} = \mathbf{M}(\mathbf{x} - \mathbf{t}) + \mathbf{t}$ to obtain

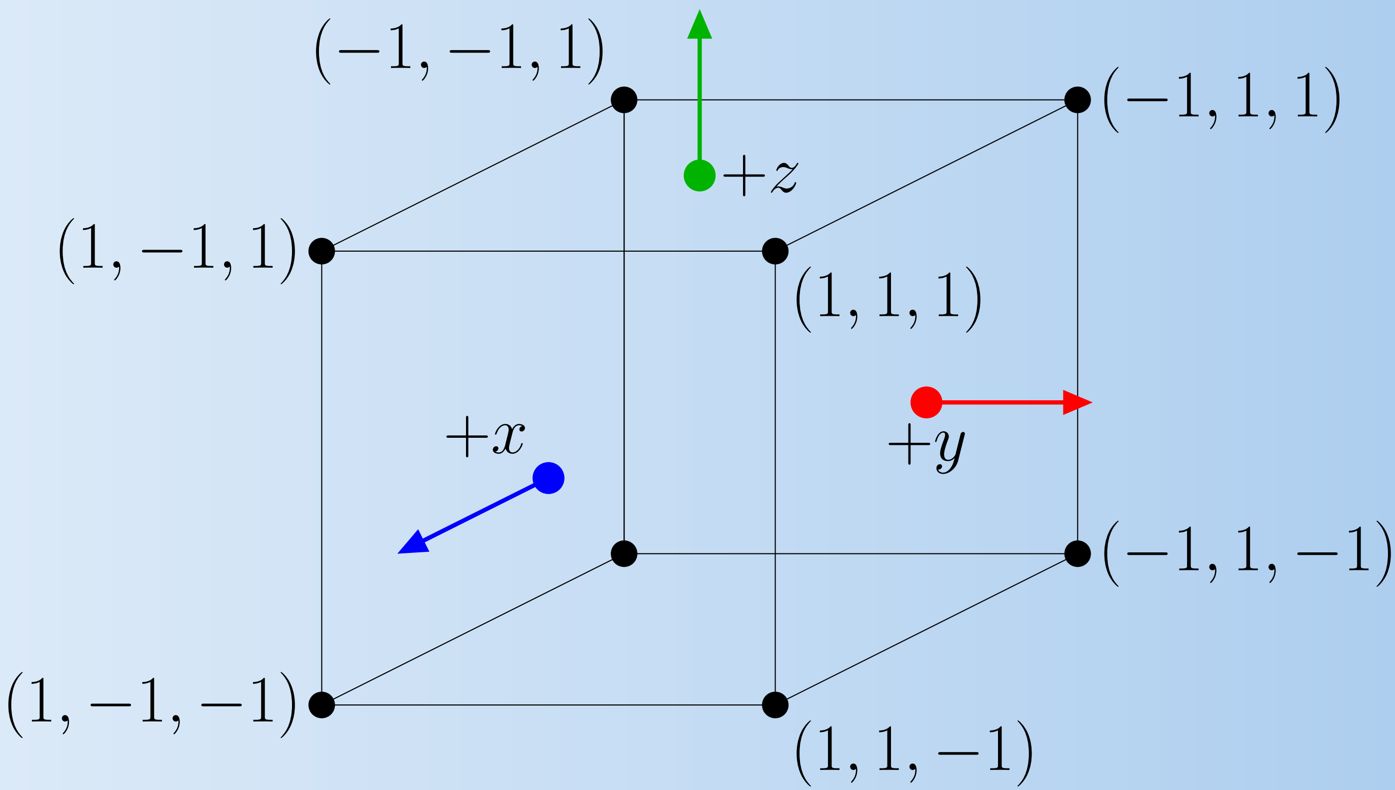
$$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{pmatrix} x \\ y-1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\frac{4}{5} \\ \frac{8}{5} \end{pmatrix}.$$

Representing a three-dimensional coordinate system in two dimensions:



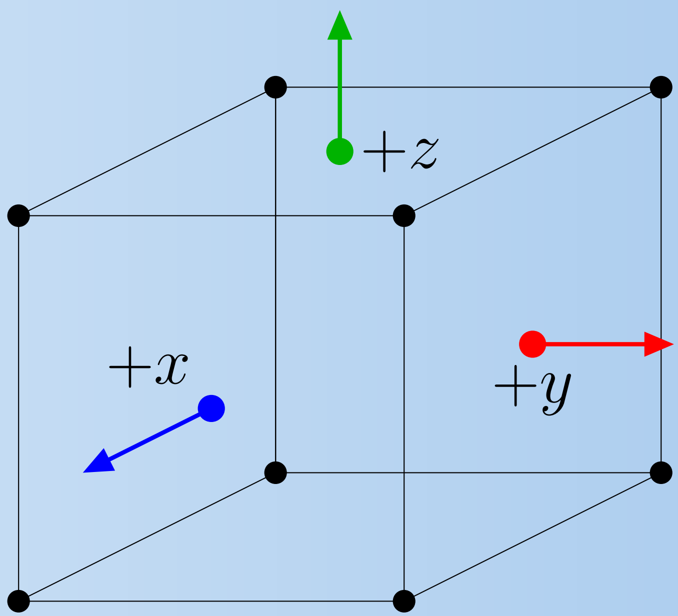
Coordinates of a cube in three dimensions:



Rotate 90° clockwise about the y-axis:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}-1\\0\\0\end{pmatrix}$$

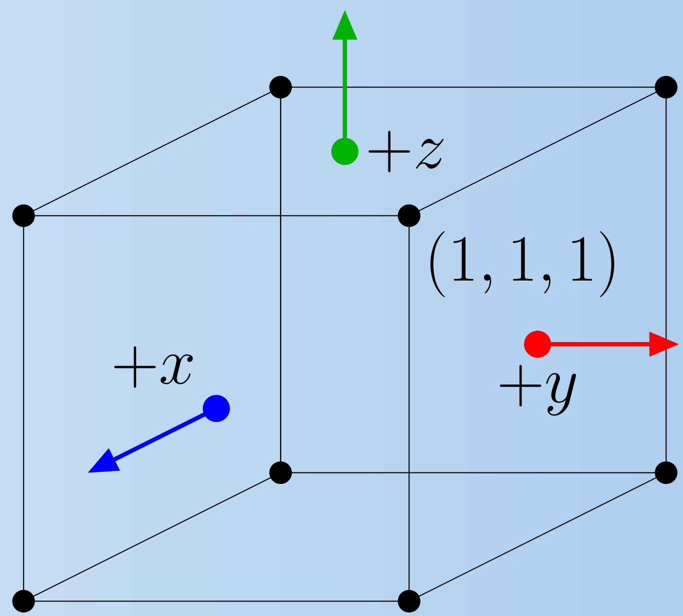
$$\mathbf{R} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$



Rotate 120° clockwise about $(1,1,1)$:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\1\\0\end{pmatrix}$$

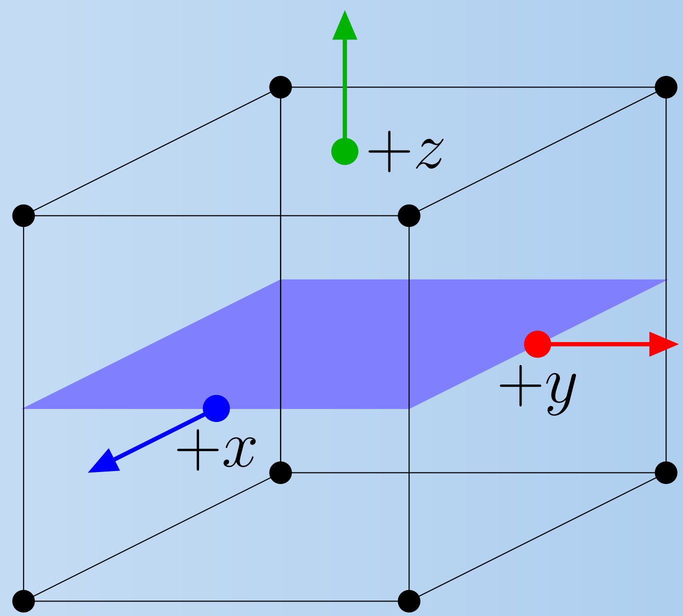
$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$



Reflecting across the xy-plane:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\-1\end{pmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

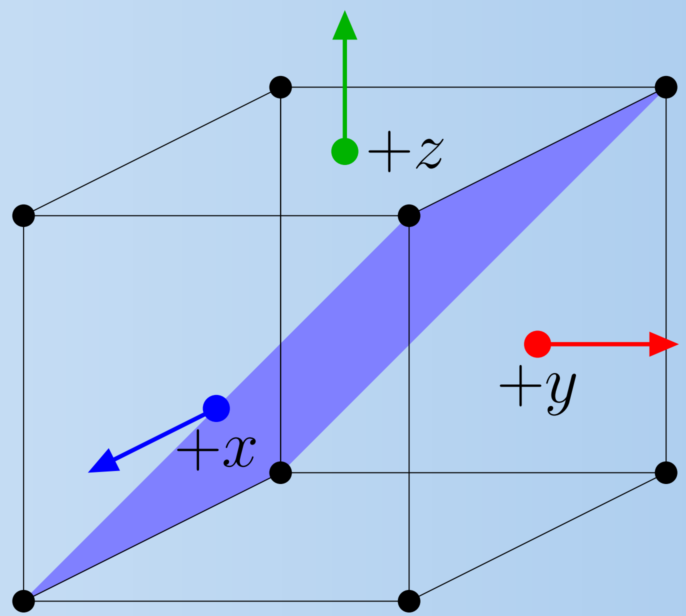


Reflecting across a Plane of Symmetry

Reflecting across another plane of symmetry:

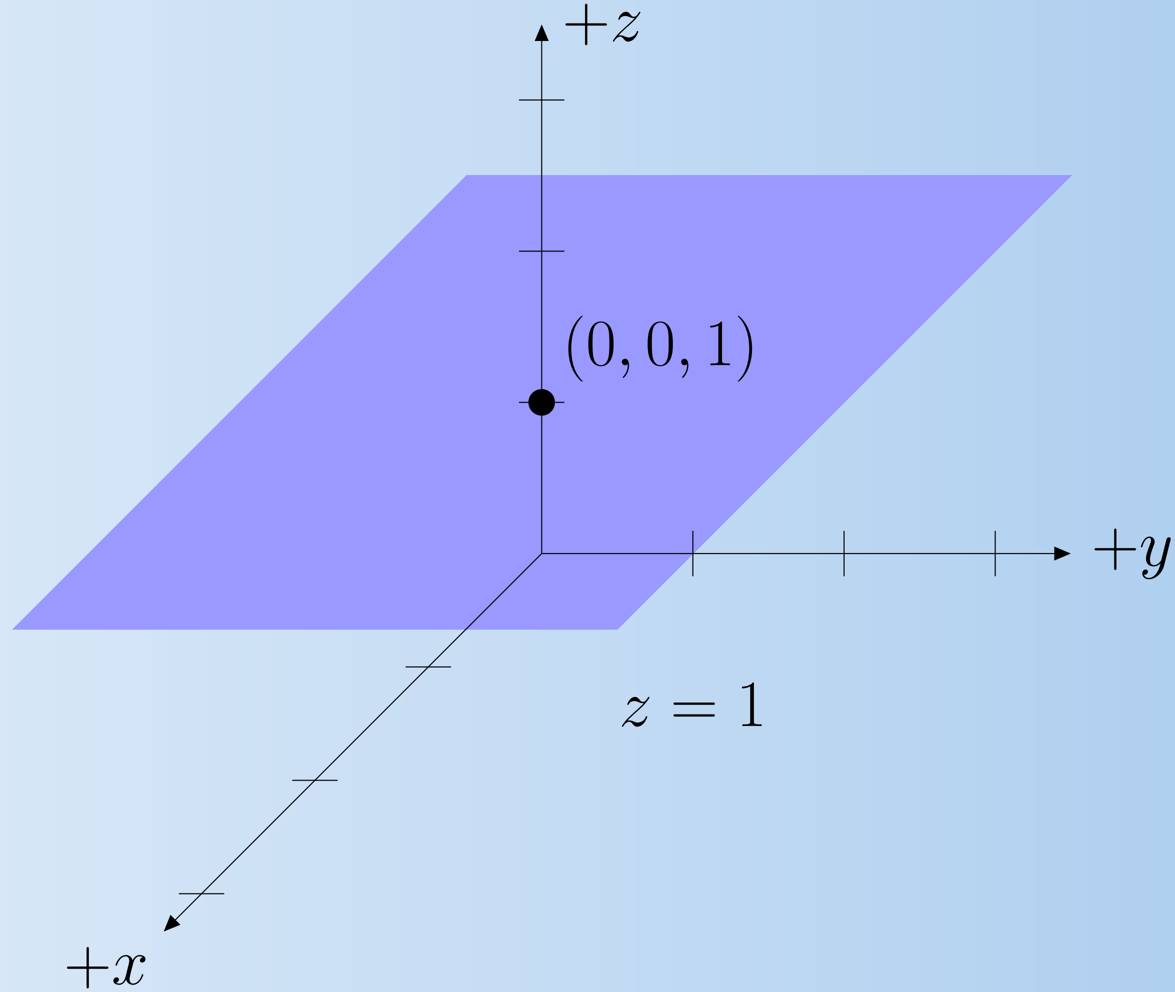
$$\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\mathbf{R} = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}$$



The xy-plane in Three Dimensions

We may embed the xy-plane in three dimensions, representing it as the plane $z = 1$.



Translations as Matrices in Three Dimensions

Recall that a translation cannot be represented by a matrix in two dimensions as translations are not linear transformations. However,

$$\begin{bmatrix} 1 & 0 & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+e \\ y+f \\ 1 \end{pmatrix}.$$

Thus, representing the xy-plane as the plane $z = 1$ in three dimensions allows a translation to be represented by a matrix.

Affine Translations with Matrices

Recall a previous transformation:

$$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{8}{5} \\ \frac{4}{5} \end{pmatrix}.$$

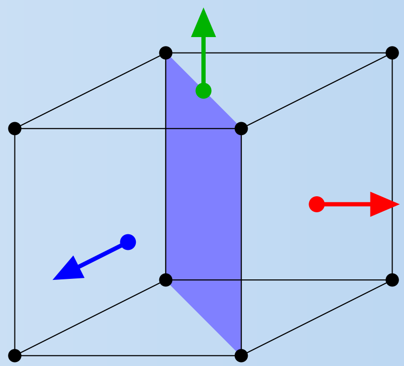
Now:

$$\begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} & \frac{8}{5} \\ -\frac{4}{5} & \frac{3}{5} & \frac{4}{5} \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5}x - \frac{4}{5}y + \frac{8}{5} \\ -\frac{4}{5}x + \frac{3}{5}y + \frac{4}{5} \\ 1 \end{pmatrix}.$$

Three-Dimensional Summary

- Coordinate representations extend to three dimensions.
- Matrices for symmetries are created as in two dimensions. Creating symmetry matrices can develop spatial visualization skills.
- Translations in two dimensions may be represented as matrices in three dimensions. This is a common application in computer graphics (four-dimensional matrices are needed!).
- Thus, any affine transformation in two dimensions can be represented as a matrix in three dimensions.

1. Find the matrix which describes rotating the cube 120° around the vertex $(1, -1, 1)$.
2. Find the matrix which describes the reflection about the following plane of symmetry:



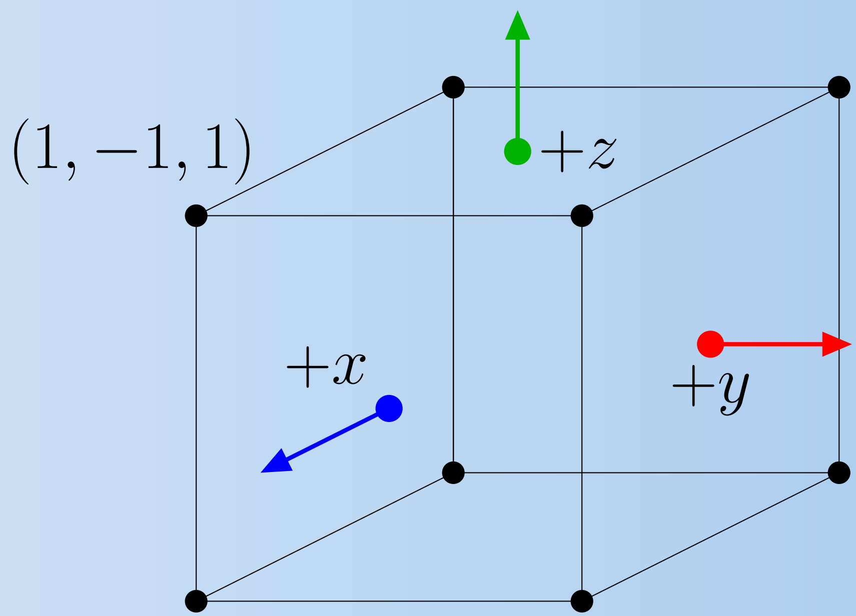
3. Find the matrix in three dimensions which represents the affine transformation

$$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$$

Rotate 120° clockwise about $(1, -1, 1)$:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}0\\-1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}0\\0\\-1\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}1\\0\\0\end{pmatrix}$$

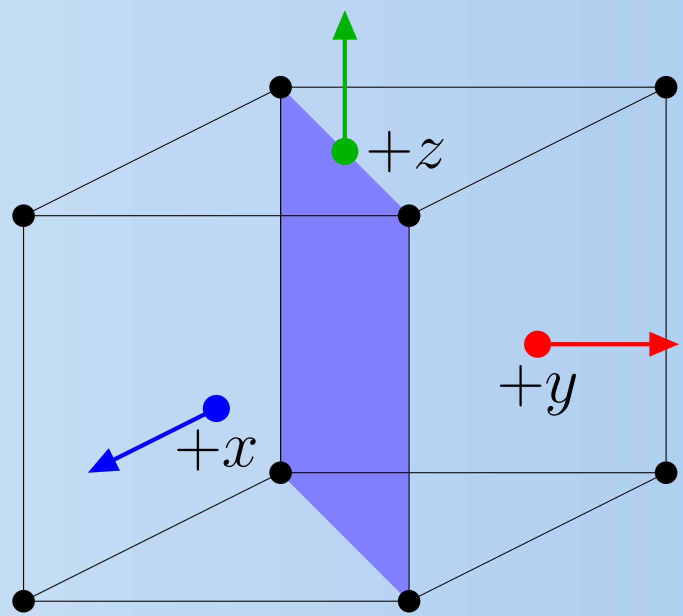
$$\mathbf{R} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$



Reflecting across the given plane of symmetry:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Find the matrix in three dimensions which represents the affine transformation

$$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$$

In three dimensions, we represent this matrix as

$$\mathbf{A}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$