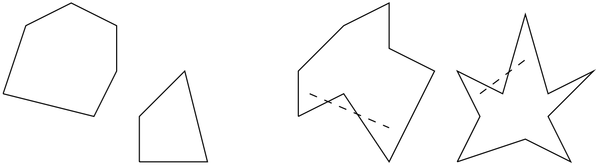
The Platonic Solids: Three Proofs

Convex Polygons



convex polygons nonconvex polygons

The Platonic Solids

A polyhedron is a Platonic solid if:

(P_1) *It is convex.*

(P_2) *Its faces are all the same regular polygon.*

(P_3) *The same number of polygons meet at each of its vertices.*

Geometrical Enumeration

Three triangles per vertex: *Tetrahedron*

Geometrical Enumeration

Three triangles per vertex: *Tetrahedron*

Four triangles per vertex: *Octahedron*

Geometrical Enumeration

Three triangles per vertex: *Tetrahedron*

Four triangles per vertex: *Octahedron*

Five triangles per vertex: *Icosahedron*

Geometrical Enumeration

Three triangles per vertex: *Tetrahedron*

Four triangles per vertex: *Octahedron*

Five triangles per vertex: *Icosahedron*

Six triangles per vertex: Flat

Geometrical Enumeration

Three triangles per vertex: *Tetrahedron*

Four triangles per vertex: *Octahedron*

Five triangles per vertex: *Icosahedron*

Six triangles per vertex: Flat

Three squares per vertex: *Cube*

Geometrical Enumeration

Three triangles per vertex: *Tetrahedron*

Four triangles per vertex: *Octahedron*

Five triangles per vertex: *Icosahedron*

Six triangles per vertex: Flat

Three squares per vertex: *Cube*

Four squares per vertex: Flat

Geometrical Enumeration

Three triangles per vertex: *Tetrahedron*

Four triangles per vertex: *Octahedron*

Five triangles per vertex: *Icosahedron*

Six triangles per vertex: Flat

Three squares per vertex: *Cube*

Four squares per vertex: Flat

Three pentagons per vertex: *Dodecahedron*

Geometrical Enumeration

Three triangles per vertex: *Tetrahedron*

Four triangles per vertex: *Octahedron*

Five triangles per vertex: *Icosahedron*

Six triangles per vertex: Flat

Three squares per vertex: *Cube*

Four squares per vertex: Flat

Three pentagons per vertex: *Dodecahedron*

Four pentagons per vertex: Overlap

Geometrical Enumeration

Three triangles per vertex: *Tetrahedron*

Four triangles per vertex: *Octahedron*

Five triangles per vertex: *Icosahedron*

Six triangles per vertex: Flat

Three squares per vertex: *Cube*

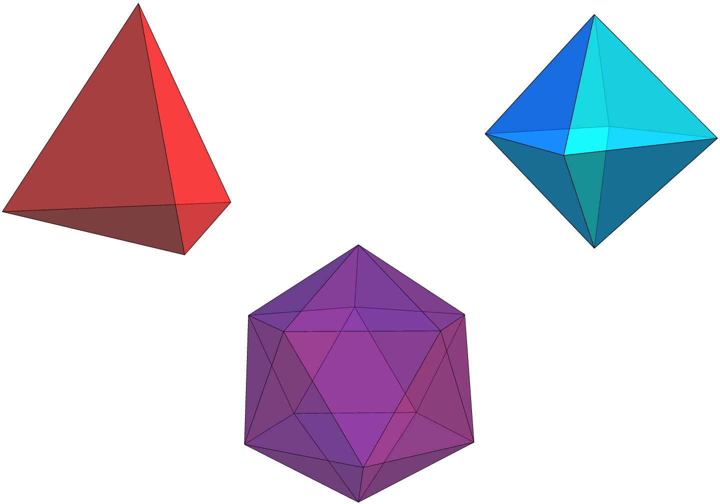
Four squares per vertex: Flat

Three pentagons per vertex: *Dodecahedron*

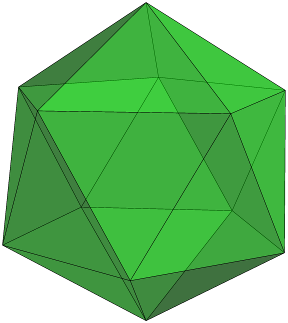
Four pentagons per vertex: Overlap

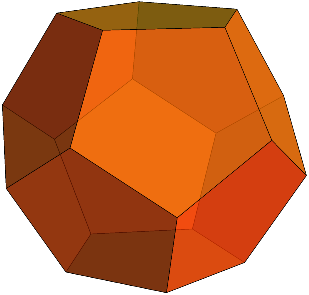
Three hexagons per vertex: Flat

The Platonic Solids (1)



Platonic Solids (2)





Euler's Formula

If V denotes the number of vertices of a polyhedron, E the number of edges, and F the number of faces, then

$$V - E + F = 2.$$

Example (the Cube):

$$8 - 12 + 6 = 2.$$

First Algebraic Enumeration (1)

Let p be the number of edges on each face and q be the number of faces meeting at each vertex.

Euler's formula is valid:

$$(P_1') \qquad V - E + F = 2.$$

Then pF counts the edges twice:

$$(P_2') \qquad pF = 2E.$$

Also, qV counts the edges twice:

$$(P_3') \qquad qV = 2E.$$

First Algebraic Enumeration (2)

Note that

$$pF = 2E \implies F = \frac{2E}{p}$$

and

$$qV = 2E \implies V = \frac{2E}{q}.$$

Substituting into Euler's formula gives

$$\frac{2E}{q} - E + \frac{2E}{p} = 2,$$

or

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E}.$$

First Algebraic Enumeration (3)

If both $p \geq 4$ and $q \geq 4$, then

$$\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}.$$

This contradicts

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E} > \frac{1}{2}.$$

Thus, either $p = 3$ or $q = 3$.

First Algebraic Enumeration (4)

Recall that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E}.$$

First, consider the case $p = 3$. We get the following table for q:

p	q	E
3	3	6
3	4	12
3	5	30

If $q \geq 6$, then $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{3} + \frac{1}{6} \leq \frac{1}{2}$, again a contradiction.

First Algebraic Enumeration (5)

Also considering the case when $q = 3$, we obtain the following five solutions, each corresponding to a different Platonic solid:

p	q	E	V	F	Platonic Solid	$\{p, q\}$
3	3	6	4	4	Tetrahedron	$\{3, 3\}$
3	4	12	6	8	Octahedron	$\{3, 4\}$
4	3	12	8	6	Cube	$\{4, 3\}$
3	5	30	12	20	Icosahedron	$\{3, 5\}$
5	3	30	20	12	Dodecahedron	$\{5, 3\}$

Second Algebraic Enumeration (1)

We may obtain this table in another way. We begin with

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E} > \frac{1}{2},$$

and so

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{2}.$$

Multiplying through by $2pq$ gives

$$2q + 2p > pq,$$

which may be rearranged as

$$(p-2)(q-2) < 4.$$

Second Algebraic Enumeration (2)

Now we have

$$(p-2)(q-2) < 4.$$

If both $p \geq 4$ and $q \geq 4$, then

$$p-2 \geq 2 \quad \text{and} \quad q-2 \geq 2,$$

and so

$$(p-2)(q-2) \geq 4,$$

a contradiction. We proceed as before: either $p = 3$ or $q = 3$ (or perhaps both).